Applications of Malliavin-Stein Method: Spatial averages of solution to stochastic heat equation and Breuer-Major theorem

Şefika Kuzgun

M.S. in Mathematics, Boğaziçi Üniversitesi, 2016

M.A. in Mathematics, University of Kansas, 2018

Submitted to the graduate degree program in the Department of Mathematics and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Committee members

David Nualart, Chairperson

Jin Feng

Mathew Johnson

Zhipeng Liu

Tarun Sabarwal, Economics

Date defended: May 4, 2022

The Dissertation Committee for Şefika Kuzgun certifies
that this is the approved version of the following dissertation :

Applications of Malliavin-Stein Method:
Spatial averages of solution to stochastic heat equation and Breuer-Major theorem

____
David Nualart, Chairperson

Date approved: ______May 4, 2022______

Abstract

This thesis includes four main parts. The first part is an exposition about Malliavin calculus, Malliavin-Stein method, Walsh stochastic integral and existence and regularity of mild solution to stochastic heat equation. In the second part, we study Malliavin differentibility of the solution of stochastic heat equation and establishing L^p-bounds for Malliavin derivatives. One way to obtain such results is through Feynman-Kac formula which is studied in the second part as well.

Last two parts are devoted to quantitative rates of convergences corresponding to some central limit theorems: We start with studying such problem for spatial averages of the solution to the stochastic heat equation. Then, we establish rate of convergence results in total variation as well as in Wasserstein distances for the Breuer-Major theorem.

Acknowledgements

I would like to thank my advisor David Nualart for his guidance throughout my studies and his patience with my questions. His enthusiasm and seriousness towards mathematics inspired me. I am grateful to Jin Feng, Mathew Johnson, Zhipeng Liu, and Tarun Sabarwal who do me the honor of being in the committee of my thesis defense. I want to express my deep gratitude to Jin Feng for valuable discussions and words of encouragement.

I would like to thank all my teachers who touched my life from primary school to university and helped me reach this point in my education. I would like to thank all my professors at University of Kansas and at Boğaziçi University. I am especially indebted to my master's thesis advisor Tuğrul Burak Gürel, my undergraduate research advisor Nihat Sadık Değer and Atilla Yılmaz for their valuable guidance. I would like to thank Rodolfo Torres for his kindness and contributions to my studies. I thank Agnieszka Międlar and Paul Cazeaux for their warm friendship and valuable advice. I thank Jila Niknejad for her friendly leadership and support.

I would like to thank fellow graduate students Ray Zhang, Chen Ma, Panqiu Xia, Josué Knorst, Raul Balaños Guerrero, Bhargobjyoti Saikia for beneficial reading sessions. I am grateful to Guangqu Zheng for his valuable insights and support. I would like to thank all fellow graduate students and young faculty at University of Kansas for their friendship.

Thank you from bottom of my heart to my big family in Turkey and in Kansas for always being there for me. In particular, I am so grateful to Melek Erkoç for her support and encouragement. I would like to thank my friend Feride Ceren Köse with whom I have shared moments of deep anxiety but also of big excitement. Finally, I thank my husband Mehmet Yenisey for his everlasting support and tirelessly proofreading my writings.

To my father Fatih, my mother Meryem, and my brother Muhammed...

Contents

Chapter 1

Introduction

In this introduction we explain the problems considered in the projects which form this thesis.

The connection between heat flow and Brownian motion is a well-known and recurring theme in the mathematical study of these two objects. Feynman-Kac formula exhibits this well-known connection by representing the solution to the heat equation with a deterministic external forcing term as an expectation of a functional of Brownian motion. One should then ask whether a similar representation exists in the case of stochastic heat equation (5.1). Indeed, in the case $g(x) = x$, see for example [25], a Feynman-Kac representation for the p-th moment of the solution is obtained as in (5.17) as a functional of $\{B_j\}_{j=1,\cdots,p}$, which is a family of independent Brownian motions independent of the noise W in the equation. The main purpose of the work in [32] which we recall in subsection 5.2.1 is to obtain a similar representation for moments of iterated derivatives $D_{r_1,z_1}\cdots D_{r_N,z_N}u(t,x)$ of the solution $u(t,x)$ in terms of independent pinned Brownian motions starting from x with each component pinned at times $t - r_m$ to the points z_m for $1 \le m \le N$. We proposed a formula Theorem 5.18 for the moments of the iterated Malliavin derivatives, which is interesting on its own, and implies the estimate Corollary 5.19 which can be immediately used together with Malliavin-Stein estimates.

Following the ideas by Conus, Joseph, and Khoshnevisan [19], Huang, Nualart and Viitasaari [27] observed that the spatial integral, $\int_{-R}^{R} u(t,x)dx$ of the solution to the equation (5.1) with the constant initial condition behaves like a sum of i.i.d. random variables. Indeed, they proved that the variance of the spatial integral behaves like R and $\int_{-R}^{R} u(t,x)dx/\sqrt{R}$ converges in distribution to a normal random variable. Using Malliavin-Stein bound Theorem 3.10, they also established a quantitative version, see Theorem 6.2. In [31] which we present in chapter 6, we studied such quantitative estimates using the distance between densities with respect to the supremum norm. First, we have established the existence of the density using results from Malliavin calculus, see Proposition 2.50. Then, we have applied a Malliavin-Stein approach to obtain Malliavin-Stein bound Theorem 3.13 between the density of a random variable given in the form $F = \delta(V)$ and the density ϕ of a standard normal distribution and then used these results to prove Theorem 6.4. One of the main two challenging parts of this methodology was the estimation of the moments of second derivative of the solution which had not been considered before except in the case $g(x) = x$. The other challenge was to get a uniform estimate for the negative moments of $\langle DF_{R,t}, V_{R,t} \rangle_{\mathfrak{H}}$ using a non-degeneracy condition on $g(u(t,x))$. The latter parts of the proof rely highly on the positivity as well as Hölder continuity of the solution.

We also considered the case with the initial condition $u_0(x) = \delta_0(x)$ and $g(x) = x$. One important difference from the previous set-up lies in the fact that for a fixed $t > 0$, the process $\{u(t,x)\}_{x\in\mathbb{R}}$ itself is not stationary but $\{U(t,x)\}_{x\in\mathbb{R}} = \{u(t,x)/p_t(x)\}_{x\in\mathbb{R}}$ is, see [1]. An advantage is that the second derivative estimate in this case follows from the bound (6.46) as a corollary to Feynman-Kac formula that we obtained in [32]. Using again Theorem 3.13, we have obtained the rate of convergence result, see Theorem 6.8. Finally, note that negative moments of $\langle DG_{R,t}, w_{R,t} \rangle_{\mathfrak{H}}$ are not necessarily bounded so that their growth must be taken into account when estimating the rate of convergence.

In the last chapter, we consider a centered stationary Gaussian sequence of random variables $X = \{X_n\}_{n\in\mathbb{N}_0}$ defined in Definition 7.1. Breuer and Major established a normal approximation result in [8] which states that if the covariance function ρ of X safisfies the integrability condition (7.3), then the sequence F_n defined in (7.1) converges in law to the centered normal distribution. Using Dini's theorem one can show that convergence holds with respect to the Kolmogorov distance, however, determining the convergence in total variation distance is a more delicate question. For example, if g is taking values in a discrete subset of $\mathbb{R}$, then $d_{\mathrm{TV}}(F_n, N(0,\sigma^2)) = 1$ for all $n \in \mathbb{N}$. In [30], we investigated the rate of convergence in total variation and Wasserstein distances associated to this normal approximation, see Theorem 7.8, Theorem 7.10, Theorem 7.15. In this paper, chaos expansions together with Malliavin-Stein method is used to establish the rates corresponding to total variation and Wasserstein distances under a technical assumption that $A(g) \in \mathbb{D}^{1,4}$ (See (7.9)). Later, in [41], Nourdin, Nualart and Peccati obtained the same bound for total variation distance under the strictly weaker assumption that $g \in \mathbb{D}^{1,4}$ using a combination of Gebelein's inequality (see [41, Lemma 2.5]) together with Malliavin-Stein bounds. In particular, the estimate is now valid for $g(x) = |x|^p - \mathrm{E}[|Z|^p]$ for any $p \geq 1$. It is important to note that the two summands on the right hand side of (7.11) are not comparable in general but the bound still implies the convergence in total variation distance under the assumption $\|\rho\|_{l^2} < \infty$ (See Lemma 7.14).

We will first give a thorough presentation of the preliminary materials in chapter 2, chapter 3, chapter 4, chapter 5 and partly in other chapters, and then present the results which we mentioned above in parts of chapter 5, chapter 6 and chapter 7. Readers can choose to read in the order chapter 2, chapter 3, chapter 4, chapter 5, chapter 6, or chapter 2, chapter 3, chapter 7 independently.

Chapter 2

Malliavin calculus

Malliavin [34] constructed a differential calculus on the Wiener space to obtain a purely probabilistic proof of Hörmander's theorem on the existence and smoothness of densities for solutions of stochastic differential equations. Since then Malliavin calculus have found its applications in various topics. In this chapter, we first recall the basic results in this theory using the book Nualart [42], as well as the books Baudoin [4], Matsumoto and Taniguchi [35], Nualart and Nualart [43], Nourdin and Peccati [38], Sanz-Solé [52], Üstünel [55] and lecture notes Bally [3], Hairer [22], Kunze [29], Nualart [47]. The density theorem presented in the last section is first proved in Caballero, Fernández, and Nualart [9].

2.1 Introduction

Let $\mathfrak{H}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$ and associated norm $\|\cdot\|_{\mathfrak{H}} = \langle\cdot,\cdot\rangle_{\mathfrak{H}}^{1/2}$. $(\Omega,\mathfrak{F},\mathrm{P})$ is a fixed probability space.

Definition 2.1. $W = \{W(h)\}_{h\in\mathfrak{H}}$ is called *isonormal Gaussian process over* $\mathfrak{H}$ if W is a centered Gaussian family, that is a collection of jointly Gaussian random variables, defined on a probability space $(\Omega,\mathfrak{F},P)$ with covariance function $\mathrm{E}[W(h)W(g)] = \langle g,h\rangle_{\mathfrak{H}}$. Further assume $\mathfrak{F}$ is the σ-field generated by W.

Lemma 2.2. Let $\{W(h)\}_{h\in\mathfrak{H}}$ be an isonormal Gaussian process over $\mathfrak{H}$. Then the map $h\mapsto W(h)$ is a linear isometry.

Proof. Let $g,h \in \mathfrak{H}$ and $\alpha,\beta \in \mathbb{R}$, then we have

$$
\begin{aligned}
&\mathrm{E}\left[(W(\alpha h+\beta g)-\alpha W(h)-\beta W(g))^2\right] \\
&=\mathrm{E}\left[(W(\alpha h+\beta g))^2\right]+\alpha^2\mathrm{E}\left[(W(h))^2\right]+\beta^2\mathrm{E}\left[(W(g))^2\right] \\
&\qquad -2\alpha\mathrm{E}\left[W(\alpha h+\beta g)W(h)\right]-2\beta\mathrm{E}\left[W(\alpha h+\beta g)W(g)\right]-2\alpha\beta\mathrm{E}\left[W(g)W(h)\right] \\
&=\|\alpha h+\beta g\|_{\mathfrak{H}}^2+\alpha^2\|h\|_{\mathfrak{H}}^2+\beta^2\|g\|_{\mathfrak{H}}^2-2\alpha\langle\alpha h+\beta g,h\rangle_{\mathfrak{H}}-2\beta\langle\alpha h+\beta g,g\rangle_{\mathfrak{H}}-2\alpha\beta\langle h,g\rangle_{\mathfrak{H}} \\
&=0
\end{aligned}
$$

which implies that $W(\alpha h+\beta g)=\alpha W(h)+\beta W(g)$ almost surely. □

Lemma 2.3. The map $h \mapsto W(h)$ is a linear isometry from $\mathfrak{H}$ to a closed subspace of $L^2(\Omega,\mathfrak{F},\mathrm{P})$ such that for all $h \in \mathfrak{H}$, $W(h)$ is a real valued centred Gaussian random variable if and only if $W=\{W(h)\}_{h\in\mathfrak{H}}$ is an isonormal Gaussian process over $\mathfrak{H}$.

Proof. For the forward direction, it is enough to show that $\{W(h)\}_{h\in\mathfrak{H}}$ is a Gaussian family. Indeed, for any $h_1,\cdots h_n \in \mathfrak{H}$ and $\alpha_1\cdots\alpha_n \in \mathbb{R}$, using linearity, we have $\sum_{i=1}^n \alpha_i W(h_i)=W(h)$ where $h=\sum_{i=1}^n \alpha_i h_i$ is centred Gaussian random variable. Converse implication follows from Lemma 2.2. □

Proposition 2.4. There exists an isonormal Gaussian process W over $\mathfrak{H}$ for any given real separable Hilbert space $\mathfrak{H}$.

Proof. Let $\{N_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. normal random variables defined on the probability space $(\Omega,\mathfrak{F},\mathrm{P})$ and let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathfrak{H}$. For $h=\sum_{i=1}^\infty h_i e_i$ where $h_i=\langle h,e_i\rangle_{\mathfrak{H}}$, $i\in\mathbb{N}$, set $W(h):=\sum_{i=1}^\infty h_i N_i$ in $L^2(\Omega,\mathfrak{F},\mathrm{P})$. Then $W:\mathfrak{H}\to L^2(\Omega,\mathfrak{F},\mathrm{P})$ is linear and each $W(h)$ is is a centred Gaussian random variable. Moreover

$$\mathrm{E}\left[W(h)W(g)\right]=\sum_{i=1}^\infty h_i g_i=\langle h,g\rangle_{\mathfrak{H}}.$$

Then, the result follows from Lemma 2.3. □

Example 2.5. Let $\mathfrak{H} := L^2([0,1], \mathscr{B}([0,1]), m)$ where m is Lebesgue measure on $[0,1]$. Then by Proposition 2.4, there is an isonormal Gaussian process W over $L^2([0,1], \mathscr{B}([0,1]), m)$. Let $B_t := W(\mathbf{1}_{[0,t]})$. Then, for $t, s \in [0,1]$

$$\mathrm{E}[B_t B_s] = \int_0^1 \mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,s]}(r)dr = s \wedge t.$$

Moreover, given $0 \le t_0 < t_1 < \cdots < t_n = t \le 1$, the functions $\mathbf{1}_{(t_0,t_1]}, \cdots, \mathbf{1}_{(t_{n-1},t]}$ are orthogonal in $L^2([0,1], \mathscr{B}([0,1]), m)$, hence $B_{t_1} - B_{t_0} = W(\mathbf{1}_{(t_0,t_1]}), \cdots, B_t - B_{t_{n-1}} = W(\mathbf{1}_{(t_{n-1},t]})$ are uncorrelated, hence independent by being jointly Gaussian. Thus the process $(B_t)_{t\in[0,1]}$ is a Brownian motion with the filtration $\mathfrak{F}_t := \sigma(B_s : s \le t)$ if we can show that it has continuous paths. Indeed, by Kolmogorov's theorem, it can be showed that B_t has continuous modification. We will write

$$\int_0^t f(s)dB(s) := W(\mathbf{1}_{[0,t]}f)$$

and call $\int_0^t f(s)dB(s)$ the *Wiener integral* of f over $[0,t]$.

Definition 2.6. For $n \in \mathbb{N}_0$, the n-th *Hermite polynomial* $H_n(x)$ is defined as, $H_0 \equiv 1$, and for $n \ge 1$:

$$H_n(x) := \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}).$$

First few Hermite polynomials are $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = \frac{x^2-1}{2}$.

Notation 2.7. Let $\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ be the density of the standard normal distribution.

Some basic properties of Hermite polynomials are listed in the following lemma:

Lemma 2.8. Hermite polynomials satisfy the following properties:

(i) For all $t \in \mathbb{R}$, $\exp\left(tx - t^2/2\right) = \sum_{n=0}^{\infty} t^n H_n(x)$ in $L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$.

(ii) For all $n \in \mathbb{N}$, $H_n'(x) = nH_{n-1}(x)$.

(iii) For all $n \in \mathbb{N}$, $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$.

(iv) For all $n \in \mathbb{N}$, $H_n(-x) = (-1)^n H_n(x)$.

The following lemma reflects the close relation between Hermite polynomials and Gaussian random variables.

Lemma 2.9. Let M, N be standard Gaussian random variables which are jointly Gaussian. Then for $m, n \in \mathbb{N}_0$, we have

$$\mathrm{E}\left[H_n(M)H_m(N)\right] = \begin{cases} 0, & \text{if } n \neq m, \\ \frac{(\mathrm{E}[MN])^n}{n!}, & \text{if } n = m. \end{cases}$$

Definition 2.10. Let W be an isonormal Gaussian process over $\mathfrak{H}$. For each $n \in \mathbb{N}_0$, the n-th *Wiener chaos* $\mathscr{H}_n$ is the closure of the linear span of $\{H_n(W(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ in $L^2(\Omega, \mathfrak{F}, \mathrm{P})$.

Note that since $H_0 \equiv 1$, the 0-th Wiener chaos $\mathscr{H}_0$ is the set of all constants and since $H_1(x) = x$, $\mathscr{H}_1 = \{W(h) : h \in \mathfrak{H}\}$ is the 1-st Wiener chaos.

Lemma 2.11. Let W be an isonormal Gaussian process over $\mathfrak{H}$ and $\{\mathscr{H}_n\}_{n \in \mathbb{N}_0}$ be the corresponding Wiener chaos. Then for $m \neq n$, $\mathscr{H}_n$ and $\mathscr{H}_m$ are orthogonal.

Lemma 2.12. The random variables $\{e^{W(h)}\}_{h \in \mathfrak{H}}$ form a total subset of $L^2(\Omega, \mathfrak{F}, \mathrm{P})$. In other words, if $X \in L^2(\Omega, \mathfrak{F}, \mathrm{P})$ is such that $\mathrm{E}\left[Xe^{W(h)}\right] = 0$ for all $h \in \mathfrak{H}$, then $X = 0$.

Theorem 2.13. Let W be an isonormal Gaussian process over $\mathfrak{H}$ and $\{\mathscr{H}_n\}_{n \in \mathbb{N}}$ be the corresponding Wiener chaos. Then

$$\bigoplus_{n=0}^{\infty} \mathscr{H}_n = L^2(\Omega, \mathfrak{F}, \mathrm{P}) \tag{2.1}$$

and this decomposition is orthogonal. In other words, every $F \in L^2(\Omega, \mathfrak{F}, \mathrm{P})$ admits a unique expansion of the form

$$F = \sum_{n=0}^{\infty} V_n \quad \text{in } L^2(\Omega, \mathfrak{F}, \mathrm{P})$$

where for each $n \in \mathbb{N}_0$, $V_n \in \mathscr{H}_n$, and $F_0 = \mathrm{E}[F]$.

Corollary 2.14. $\{\sqrt{n!}H_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$.

Proof. Let $(\Omega, \mathfrak{F}, \mathrm{P}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ and $\mathfrak{H} = \mathbb{R}$. Define $W : \mathbb{R} \to L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ by $(W(h))(x) = hx$. Then W is an isonormal Gaussian process. Indeed, under $\phi(x)dx$, x is a Gaussian random variable. Moreover, $\mathrm{E}[W(h)W(g)] = hg\int_{\mathbb{R}} x^2 d\phi(x)dx = hg$. Furthermore, note that $\mathfrak{H} = \mathbb{R}$ has only two elements of norm 1 which corresponds to the random variables x and $-x$. But since $H_n(-x) = (-1)^n H_n(x)$ from Lemma 2.8, each $\mathscr{H}_n$ is one-dimensional. Thus, by Theorem 2.13 and Lemma 2.9, $\{\sqrt{n!}H_n\}$ is an orthonormal basis of $L^2(\mathbb{R}, \mathfrak{F}, \phi(x)dx)$. Finally, claim follows by noting that $\mathfrak{F} = \sigma(x) = \mathscr{B}(\mathbb{R})$. □

Definition 2.15. Let $f \in L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ have mean zero. By Corollary 2.14, f admits Hermite expansion

$$f(x) = \sum_{n=1}^{\infty} a_n H_n(x).$$

The *Hermite rank* of the function f is then defined as

$$\inf\{n \geq 1 : a_1 = a_2 = \cdots = a_{n-1} = 0, a_n \neq 0\} =: d.$$

2.2 Malliavin derivative

Let $C_p^\infty(\mathbb{R}^m)$ denote the set of all infinitely continuously differentiable functions $f : \mathbb{R}^m \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. Let $S = \cup_{m\in\mathbb{N}} C_p^\infty(\mathbb{R}^m)$. Let $\mathscr{S}$ denote the set of all random variables of the form $f(W(h_1), \cdots, W(h_m))$, where $m \geq 1$, $f \in C_p(\mathbb{R}^m)$ and $h_i \in \mathfrak{H}$, for $i = 1, \cdots, m$. Elements of $\mathscr{S}$ will be called smooth functionals of W. Also for any separable Hilbert space $\mathfrak{K}$, set

$$\mathscr{S}(\mathfrak{K}) := \left\{ \sum_{j=1}^{n} F_j k_j : F_j \in \mathscr{S},\ k_j \in \mathscr{K},\ j = 1, \cdots n,\ n \in \mathbb{Z}_+ \right\}.$$

Lemma 2.16. The spaces $\mathscr{S}$ and $\mathscr{S}(\mathfrak{K})$ are dense in $L^p(\Omega,\mathfrak{F},\mathrm{P})$, $L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{K})$ respectively for every $p\in[1,\infty)$.

For $p>1$, this claim can be proved by showing for all $X\in L^{\frac{p}{p-1}}$, $\mathrm{E}[XF]=0$ for all $F\in\mathscr{S}$ implies $X=0$ a.e.

Definition 2.17. Let $F\in\mathcal{S}$ be of the form $f(W(h_1),\cdots,W(h_m))$ for some $h_1\cdots h_m\in\mathfrak{H}$ and $m\in\mathbb{N}_0$. The *Malliavin Derivative DF* of F (with respect to the underlying isonormal Gaussian family W) is the element of $L^2(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})$ defined by

$$DF:=\sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(W(h_1),\cdots,W(h_m))h_i. \tag{2.2}$$

Remark 2.18. This definition is well-defined in the sense that it doesn't depend on the representation of the given random variable. To see this let $\{e_i\}_{i\in\mathbb{N}}\subset\mathfrak{H}$ be an orthonormal basis and $h_1,\cdots h_m\in\mathfrak{H}$. Assume $F\in\mathscr{S}$ has representations

$$F=f(W(h_1),\cdots,W(h_m))=g(W(e_1),\cdots,W(e_n))$$

for some f,g. Without loss of generality we may assume that

$$\mathrm{span}\{e_1,\cdots,e_n\}=\mathrm{span}\{h_1,\cdots,h_m\},$$

and $m=n$. Otherwise, we can let $h_{m+1}=e_1,\cdots,h_{n+m}=e_n$ and $e_{m+1}=h_1,\cdots,e_{m+n}=h_m$ and replacing f,g with $\tilde{f},\tilde{g}$, where $\tilde{f}(x_1,\cdots,x_{m+n})=f(x_1,\cdots,x_n)$ and $\tilde{g}(x_1,\cdots,x_{m+n})=g(x_1,\cdots,x_m)$. This doesn't effect the derivative because $\partial_j\tilde{f}=0$ for $j>n$ and $\partial_j\tilde{g}=0$ for $j>n$. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be the linear transformation such that $T_{ij}=\langle h_i,e_j\rangle$ for all $i,j\in\{1,\cdots,n\}$. Then by linearity of W, $T(W(e_1),\cdots W(e_n))=(W(h_1),\cdots,W(h_n))$, so that

$$(f\circ T)(W(e_1),\cdots W(e_n))=g(W(h_1),\cdots,W(h_n))=X. \tag{2.3}$$

This implies that $f \circ T = g$. Indeed, if $f \circ T(x_0) \neq g(x_0)$ for some for some $x_0 \in \mathbb{R}^n$, then by continuity $|f \circ T - g| > \varepsilon$ in a neighborhood of x_0. Since the standard Gaussian vector $W(e_1), \cdots W(e_n)$ has strictly positive probability of being in that neighbourhood, this contradicts the equality (2.3). This using the chain rule from elementary calculus,

$$\begin{aligned}\sum_{i=1}^{n} \partial_i g(W(e_1), \cdots W(e_n)) e_i &= \sum_{i=1}^{n} \partial_i f \circ T(W(e_1), \cdots W(e_n)) e_i \\ &= \sum_{i,j=1}^{n} \partial_j (f \circ T)(W(e_1), \cdots W(e_n)) \langle h_j, e_i \rangle e_i \\ &= \sum_{j=1}^{n} \partial_j f(W(h_1), \cdots W(h_n)) h_j.\end{aligned}$$

Before we get into some properties of the Malliavin operator, let us consider some examples.

Example 2.19. If $f(x) = x$, we see $D(W(h)) = h$.

Example 2.20. Let W be as in Example 2.5 and $F = f(W(1_{[0,t]})) \in \mathscr{S}$. Then for each $h \in \mathfrak{H} = L^2([0,1], \mathscr{B}([0,1]), m)$, using the definition of Malliavin derivative, we have

$$\langle DF, h \rangle_{\mathfrak{H}} = f'(W(1_{[0,t]})) \langle 1_{[0,t]}, h \rangle_{\mathfrak{H}} = f'(W(1_{[0,t]})) \int_0^t h(s) ds.$$

Note that the left hand side of this equation in the path space is also equal to

$$\frac{d}{d\varepsilon} F(\omega + \varepsilon \int_0^{\cdot} h(s) ds)\Big|_{\varepsilon=0}.$$

Define the the *Cameron-Martin space* $\mathfrak{H}^1$ of Ω as

$$\mathfrak{H}^1 := \{\tilde{h} \in C([0,1]) : \tilde{h}(t) = \int_0^t h(s) ds, \text{ for some } h \in \mathfrak{H}\}.$$

$\mathfrak{H}^1$ is an Hilbert space with the inner product

$$\langle \tilde{h}, \tilde{g} \rangle_{\mathfrak{H}^1} = \int_0^1 h(s) g(s) ds,$$

and it is isomorphic to $\mathfrak{H}$. Then, for any $h \in \mathfrak{H}$, $\langle DF, h \rangle_{\mathfrak{H}}$ is the directional derivative of F in the direction $\tilde{h} \in \mathfrak{H}^1$ where $\tilde{h}(t) = \int_0^t h(s)ds$.

Remark 2.21. In general, the derivative DF can be interpreted as the directional derivative as follows: For $F = f(W(h)) \in \mathscr{S}$ and $g \in \mathfrak{H}$, on one had, we have

$$\langle DF, g \rangle_{\mathfrak{H}} = f'(W(h))\langle h, g \rangle_{\mathfrak{H}},$$

and on the other hand

$$\lim_{\varepsilon \to 0} \frac{f(W(h) + \varepsilon \langle h, g \rangle_{\mathfrak{H}}) - f(W(h))}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{f'(W(h))\varepsilon \langle h, g \rangle_{\mathfrak{H}}}{\varepsilon} = f'(W(h))\langle h, g \rangle_{\mathfrak{H}}.$$

Hence, one has

$$\langle DF, g \rangle_{\mathfrak{H}} = \lim_{\varepsilon \to 0} \frac{f(W(h) + \varepsilon \langle h, g \rangle_{\mathfrak{H}}) - f(W(h))}{\varepsilon}.$$

Now we will prove some preliminary integration by parts formula which will then allow us to extend the derivative operator to a larger class of random variables.

Lemma 2.22. Let $F, \tilde{F} \in \mathscr{S}$ and $h \in \mathfrak{H}$. Then

$$\mathrm{E}\left[\langle DF, h \rangle_{\mathfrak{H}}\right] = \mathrm{E}\left[FW(h)\right], \tag{2.4}$$

$$\mathrm{E}\left[\tilde{F}\langle DF, h \rangle_{\mathfrak{H}}\right] = \mathrm{E}\left[F\tilde{F}W(h)\right] - \mathrm{E}\left[F\langle D\tilde{F}, h \rangle_{\mathfrak{H}}\right]. \tag{2.5}$$

Proof. Note that Leibniz formula

$$D(F\tilde{F}) = \tilde{F}DF + FD\tilde{F} \tag{2.6}$$

follows from the Leibniz formula for the usual derivative. For (2.4) we may assume $\|h\|_{\mathfrak{H}} = 1$ by linearity and $F = f(W(e_1), W(e_2), \cdots, W(e_n))$ where $f \in S$ and $\{e_1, e_2, \cdots, e_n\} \subset \mathfrak{H}$ are orthonormal and $e_1 = h$. Then using the usual integration by parts, we get

$$\mathrm{E}\left[\langle DF, h\rangle_{\mathfrak{H}}\right] = \int_{\mathbb{R}^n} \partial_1 f(x)\phi(x)dx = \int_{\mathbb{R}^n} f(x)\phi(x)x_1 dx = \mathrm{E}\left[FW(e_1)\right] = \mathrm{E}\left[FW(h)\right].$$

Notice if $F, \tilde{F} \in \mathscr{S}$ so is $F\tilde{F}$. Applying (2.4) to $F\tilde{F}$ and using (2.6), we finally obtain (2.5). $\square$

Proposition 2.23. Let $p \in [1, \infty)$. Then the operator $D : \mathscr{S} \subset L^p(\Omega, \mathfrak{F}, \mathrm{P}) \to L^p(\Omega, \mathfrak{F}, \mathrm{P}; \mathfrak{H})$ is closable. In other words for every sequence $\{F_n\}_{n\in\mathbb{N}_0} \subset \mathscr{S}$ such that $F_n \to 0$ in $L^p(\Omega, \mathfrak{F}, \mathrm{P})$ as $n \to \infty$, and $DF_n \to Y$ in $L^p(\Omega, \mathfrak{F}, \mathrm{P}; \mathfrak{H})$ as $n \to \infty$, it holds that $Y = 0$ P-a.e.

Proof. We will give a proof for the case $p > 1$. Let $\{F_n\}_{n\in\mathbb{N}_0}$ be a sequence in $\subset \mathscr{S}$ such that $F_n \to 0$ in $L^p(\Omega, \mathfrak{F}, \mathrm{P})$ as $n \to \infty$, and $DF_n \to Y$ in $L^p(\Omega, \mathfrak{F}, \mathrm{P}; \mathfrak{H})$ as $n \to \infty$. Then $\langle DF_n, h\rangle_{\mathfrak{H}} \to \langle Y, h\rangle_{\mathfrak{H}}$ in $L^p(\Omega, \mathfrak{F}, \mathrm{P})$ for any $h \in \mathfrak{H}$. Let $G \in \mathscr{S}$. By (2.5), we have

$$\mathrm{E}\left[G\langle Y, h\rangle_{\mathfrak{H}}\right] = \lim_{n\to\infty} \mathrm{E}\left[G\langle DF_n, h\rangle_{\mathfrak{H}}\right] = \lim_{n\to\infty} \mathrm{E}\left[F_n\left(W(h)G - \langle DG, h\rangle_{\mathfrak{H}}\right)\right] = 0$$

where the last equality follows from Hölder's inequality since $F_n \to 0$ in $L^p(\Omega, \mathfrak{F}, \mathrm{P})$ and $W(h)G$, $\langle DG, h\rangle_{\mathfrak{H}} \in L^{\frac{p}{p-1}}$. Now since $\mathrm{E}\left[G\langle Y, h\rangle_{\mathfrak{H}}\right] = 0$ for all $G \in \mathscr{S}$, by Lemma 2.16, we have, for all $h \in \mathfrak{H}$, $\langle Y, h\rangle_{\mathfrak{H}} = 0$ P-a.e. which then implies $Y = 0$, P-a.e. $\square$

We will use the same notation for the closed extension of the derivative. Fix $p \in [1, \infty)$, the domain of the operator D is the space $\mathbb{D}^{1,p}$, defined as the closure of $\mathcal{S}$ with respect to the norm:

$$\|F\|_{\mathbb{D}^{1,p}} = \left(\mathrm{E}\left[|F|^p\right] + \mathrm{E}\left[\|DF\|_{\mathfrak{H}}^p\right]\right)^{1/p}.$$

Observe that $\mathbb{D}^{1,2}$ is a Hilbert space with the inner product

$$\langle F, G\rangle_{\mathbb{D}^{1,2}} = \mathrm{E}\left[FG\right] + \mathrm{E}\left[\langle DF, DG\rangle_{\mathfrak{H}}\right].$$

More generally, we can also define the Malliavin derivative as an unbounded operator from $\mathscr{S}(\mathfrak{K}) \subset$

$L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{K})$ to $L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H}\otimes\mathfrak{K})$ as

$$DF := \sum_{j=1}^{n} DF_j \otimes k_j.$$

Consequently, we can define the k-th Malliavin derivative of F, denoted D^kF, for any $k \in \mathbb{N}$, as the $\mathfrak{H}^{\otimes k}$-valued random variable obtained by iterating k-times the operator D. That is to say,

$$D^kF = \sum_{i_1,\cdots,i_k=1}^{m} \partial^k_{i_1,\cdots,i_k} f(W(h_1),\cdots W(h_m))(h_{i_1}\otimes\cdots\otimes h_{i_k}).$$

Similar to Proposition 2.23, $D^k : \mathscr{S} \subset L^p(\Omega,\mathfrak{F},\mathrm{P}) \to L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H}^{\otimes k})$ can be shown to be closable. The domain of the operator D^k is the space $\mathbb{D}^{k,p}$ defined as the completion of $\mathscr{S}$ with respect to the norm

$$\|F\|_{\mathbb{D}^{k,p}} = \left(\sum_{i=0}^{k} E(\|D^iF\|^p_{\mathfrak{H}^{\otimes i}})\right)^{1/p}$$

where we used the convention $\mathfrak{H}^0 = \mathbb{R}$, $D^0F = F$ and $\|\cdot\|_{0,p} = \|\cdot\|_p$. We will call $\mathbb{D}^{k,p}$ the *domain* of D^k in $L^p(\Omega,\mathfrak{F},\mathrm{P})$. Finally, we set $\mathbb{D}^{\infty,p} := \cap_{k\geq 1}\mathbb{D}^{k,p}$, and $\mathbb{D}^{\infty} := \cap_{p\geq 1}\mathbb{D}^{\infty,p}$. Furthermore, for any other separable Hilbert space $\mathfrak{K}$, let $\mathbb{D}^{k,p}(\mathfrak{K})$ denote the domain of D^k viewed as an unbounded operator from $L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{K})$ to $L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H}^{\otimes k}\otimes\mathfrak{K})$.

Proposition 2.24. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Suppose $F \in \mathbb{D}^{1,p}$ for some $p \geq 1$. Then $\varphi(F) \in \mathbb{D}^{1,p}$ and

$$D(\varphi(F)) = \varphi'(F)DF.$$

Proof. If $F \in \mathscr{S}$, this result easily follows from classical chain rule. In general, let $\{F_n\}_{n\in\mathbb{N}} \subset \mathscr{S}$ be a sequence converging to F in $\mathbb{D}^{1,p}$. In other words, $F_n = f_n(W(h_1),\cdots,W(h_{m_n})) \to F$ in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ and $DF_n \to DF$ in $L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})$. Further assume $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_b^\infty$ be a sequence of bounded functions such that $\varphi_n(x) \to \varphi(x)$ pointwise. (Existence of such sequence can be verified using mollifiers.) Then, $\varphi_n(F_n) \in \mathscr{S}$ and

$$\begin{aligned}
D(\varphi_n(F_n)) &= \sum_{i=1}^{m_n} (\varphi_n \circ f_n)'(W(h_1), \cdots, W(h_{m_n})h_i \\
&= \sum_{i=1}^{m_n} \varphi_n'(f_n(W(h_1, \cdots, h_{mn})))f_n'(W(h_1), \cdots, W(h_{m_n})h_i = \varphi_n'(F_n)DF_n.
\end{aligned}$$

Moreover, by triangle inequality, we have

$$\begin{aligned}
\left\| \varphi_n'(F_n)DF_n - \varphi'(F)DF \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})} &\leq \left\| \varphi_n'(F_n)(DF_n - DF) \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})} \\
&+ \left\| (\varphi_n'(F_n) - \varphi'(F_n))DF \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})} + \left\| (\varphi'(F_n) - \varphi'(F))DF \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})}.
\end{aligned}$$

Observe that $\sup_{n\in\mathbb{N}} |\varphi_n'(F_n)| \leq C < \infty$ a.s. and hence the first term in the right hand side of the above inequality converges to zero as $n \to \infty$. Moreover, dominated convergence theorem implies that the other two terms converge to zero as $n \to \infty$. Thus we obtain, $D(\varphi_n(F_n))$ converges to $\varphi'(F)DF$ in $L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})$ as $n \to \infty$. But on the other hand, $\varphi_n'(F_n)$ converges to $\varphi(F)$ in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ as $n \to \infty$. Finally, applying the closability of the operator D in Proposition 2.23, we get $\varphi(F) \in \mathbb{D}^{1,p}$ and $D(\varphi(F)) = \varphi'(F)DF$. For the argument where φ is Lipschitz see [42, Proposition 1.23].

□

Lemma 2.25. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathbb{D}^{1,2}$ which converges to F in $L^2(\Omega,\mathfrak{F},\mathrm{P})$ and such that

$$\sup_{n\in\mathbb{N}} \mathrm{E}\left[\|DF_n\|_{\mathfrak{H}}^2\right] < \infty.$$

Then $F \in \mathbb{D}^{1,2}$, and the sequence of derivatives $\{DF_n\}_{n\in\mathbb{N}}$ converges to DF in the weak topology of $L^2(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})$.

Lemma 2.26. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of random variables converging to F in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ for some $p > 1$. Suppose that

$$\sup_{n \in \mathbb{N}} \|F_n\|_{\mathbb{D}^{k,p}} < \infty.$$

for some $k \geq 1$. Then $F \in \mathbb{D}^{k,p}$.

2.3 Divergence operator

In this section we will introduce the adjoint of the derivative operator which is called divergence operator. (In the white noise case it is also called Skorohod integral)

Definition 2.27. We call the adjoint of the derivative operator *divergence operator* and denote it by δ. That is, δ is an unbounded operator from $\mathrm{Dom}(\delta) \subset L^2(\Omega, \mathfrak{F}, \mathrm{P}; \mathfrak{H})$ to $L^2(\Omega, \mathfrak{F}, \mathrm{P})$ such that:

- The domain of δ, denoted by $\mathrm{Dom}(\delta)$, is the subset of $L^2(\Omega, \mathfrak{F}, \mathrm{P}; \mathfrak{H})$ composed of those elements V such that there exists a $c_V > 0$ satisfying

$$|\mathrm{E}\left[\langle DF, V\rangle_{\mathfrak{H}}\right]| \leq c_V \sqrt{\mathrm{E}\left[F^2\right]} \text{ for all } F \in \mathcal{S}$$
$$\text{or, equivalently, for all } F \in \mathbb{D}^{1,2}.$$

- If $V \in \mathrm{Dom}(\delta)$, then $\delta(V)$ is the unique element of $L^2(\Omega, \mathfrak{F}, \mathrm{P})$ characterized by the following duality formula:

$$\mathrm{E}\left[F\delta(V)\right] = \mathrm{E}\left[\langle DF, V\rangle_{\mathfrak{H}}\right] \text{ for all } F \in \mathcal{S} \tag{2.7}$$
$$\text{or, equivalently, for all } F \in \mathbb{D}^{1,2}.$$

Such operator exists: fix $V \in \mathrm{Dom}(\delta)$, then the linear operator $F \to \mathrm{E}\left[\langle DF, V\rangle_{\mathfrak{H}}\right]$ is continuous from $\mathscr{S}$, equipped with the $L^2(\Omega, \mathfrak{F}, \mathrm{P})$-norm, into $\mathbb{R}$. By Riesz representation theorem, there exists a unique element in $L^2(\Omega, \mathfrak{F}, \mathrm{P})$, which we denoted by $\delta(V)$, satisfying (2.7). Some properties of this operator is listed in the following proposition.

Proposition 2.28. (i) δ is a linear and closed operator in $\operatorname{Dom}(\delta)$.

(ii) $\mathrm{E}[\delta(V)] = 0$ for all $V \in \operatorname{Dom}(\delta)$.

(iii) If $V \in \mathscr{S}(\mathfrak{H})$, then $V \in \operatorname{Dom}(\delta)$ and

$$\delta(V) = \sum_{i=1}^{m} F_i W(h_i) - \sum_{i=1}^{m} \langle DF_i, h_i \rangle_{\mathfrak{H}}.$$

(iv) Let $V \in \mathscr{S}(\mathfrak{H})$, $F \in \mathscr{S}$ and $h \in \mathfrak{H}$. Then

$$\langle D(\delta(V)), h \rangle_{\mathfrak{H}} = \langle V, h \rangle_{\mathfrak{H}} + \delta\left(\sum_{i=1}^{m} \langle DF_i, h \rangle_{\mathfrak{H}} h_i\right).$$

Proof. (i) δ is closed by being the adjoint of an unbounded densely defined operator.

(ii) This follows from applying (2.7) with $F = 1$.

(iii) Let us first show that $\mathscr{S}(\mathfrak{H}) \subset \operatorname{Dom}(\delta)$. Let $V \in \mathscr{S}(\mathfrak{H})$, then $V = \sum_{i=1}^{n} F_i h_i$ for some $F_i \in \mathscr{S}$ and $h_i \in \mathfrak{H}$ for $i = 1, \cdots n$, $n \in \mathbb{Z}_+$. Using (2.5), we have, for all $F \in \mathscr{S}$,

$$\begin{aligned} |\mathrm{E}[\langle DF, V \rangle_{\mathfrak{H}}]| &= \left|\sum_{i=1}^{n} \mathrm{E}[F_i \langle DF, h_i \rangle_{\mathfrak{H}}]\right| \\ &\leq \sum_{i=1}^{n} \left(|\mathrm{E}[F \langle DF_i, h_i \rangle_{\mathfrak{H}}]| + |\mathrm{E}[F F_i W(h_i)]|\right) \\ &\leq c_V \|F\|_{L^2(\Omega, \mathfrak{F}, \mathrm{P})}, \end{aligned}$$

where the last line follows from Cauchy-Schwarz inequality and $c_V < \infty$ follows from $F_i = f(W(h_{i_1}), \cdots, W(h_{i_j}))$ where f_i and and its derivatives has at most polynomial growth. This proves $V \in \operatorname{Dom}(\delta)$. Moreover, using (2.5), we get for all $F \in \mathscr{S}$,

$$\mathrm{E}[F\delta(V)] = \mathrm{E}[\langle DF, V \rangle_{\mathfrak{H}}] = \mathrm{E}\left[\sum_{i=1}^{n} (F_i W(h_i) - \langle DF_i, h_i \rangle_{\mathfrak{H}})\right]$$

which verifies (iii).

(iv) Using (iii) for $\delta(V)$, we get

$$\langle D(\delta(V)),h\rangle_{\mathfrak{H}}=\sum_{i=1}^{n}\left(W(h_i)\langle DF_i,h\rangle_{\mathfrak{H}}+F_i\langle h_i,h\rangle+W(h_i)\langle D(\langle DF_i,h_i\rangle_{\mathfrak{H}},h\rangle_{\mathfrak{H}})\right).$$

On the other hand, again using (2.5), we have

$$\delta\left(\sum_{i=1}^{m}\langle DF_i,h\rangle_{\mathfrak{H}}h_i\right)=\sum_{i=1}^{n}\big(W(h_j)\langle DF_i,h\rangle-\langle D\langle DF_i,h\rangle,h_i\rangle_{\mathfrak{H}}\big).$$

Then the claim follows putting these two equations together.

□

Proposition 2.29. If $V\in\mathbb{D}^{1,2}(\mathfrak{H})$, then

$$\|\delta(V)\|^2_{L^2(\Omega,\mathfrak{F},\mathrm{P})}=\mathrm{E}\left[\delta(V)^2\right]\leq\|V\|_{\mathbb{D}^{1,2}(\mathfrak{H})}.$$

In particular, $\mathbb{D}^{1,2}(\mathfrak{H})\subset\mathrm{Dom}\,\delta$ and $\delta:\mathbb{D}^{1,2}(\mathfrak{H})\to L^2(\Omega,\mathfrak{F},\mathrm{P})$ is continuous.

The following lemma is a factorization property of the divergence operator, obtained in this generality in [9, Lemma 1].

Lemma 2.30. Fix $p,p'>1$ with $1/p+1/p'=1$. Let $F\in\mathbb{D}^{1,p'}$, $V\in\mathrm{Dom}\,\delta$, be such that $V\in L^p(\Omega,\mathfrak{F},\mathrm{P};\mathfrak{H})$ and $\delta(V)\in L^p(\Omega,\mathfrak{F},\mathrm{P})$. Then $FV\in\mathrm{Dom}\,\delta$, and

$$\delta(FV)=F\delta(V)-\langle DF,V\rangle_{\mathfrak{H}}.$$

Because δ is a continuous linear operator from $\mathbb{D}^{1,p}(\mathfrak{H})$ to $L^p(\Omega,\mathfrak{F},\mathrm{P})$, Lemma 2.30 holds true provided $F\in\mathbb{D}^{1,p'}$ and $V\in\mathbb{D}^{1,p}(\mathfrak{H})$.

2.4 Wiener chaos

In this section, we will consider the case where $\mathfrak{H} = L^2(T, \mathscr{B}(T), \mu)$ for a σ-finite measure space $(T, \mathscr{B}(T), \mu)$ without atoms.

Remark 2.31. Given a random variable $F \in \mathbb{D}^{1,2}$, the Malliavin derivative DF is an element of $L^2(\Omega, \mathfrak{F}, \mathrm{P}; L^2(T, \mathscr{B}(T), \mu))$ which can be identified with $L^2(T \times \Omega, \mathscr{B}(T) \otimes \mathfrak{F}, \mu \otimes \mathrm{P})$. Thus the Malliavin derivative can be viewed as a stochastic process $\{D_t F : t \in T\}$ where $D_t F$ is defined a.e with respect to to the measure $\mu \otimes \mathrm{P}$. Similar remarks also apply to divergence operator.

Definition 2.32. Let $\{W(h)\}_{h \in \mathfrak{H}}$ be an isonormal Gaussian processes over $\mathfrak{H}$ where $\mathfrak{H} = L^2(T, \mathscr{B}, \mu)$ for a σ-finite measure space $(T, \mathscr{B}, \mu)$ without atoms. Then $W(A) := W(1_A)$ for $A \in \mathscr{B}$ is called the **white noise** on T. It has the covariance structure

$$\mathrm{E}\left[W(A)W(B)\right] = \int_T 1_A(x) 1_B(x) \mu(dx) = \mu(A \cap B).$$

Remark 2.33. Notice that in this case, we can recover the isonormal Gaussion process from the white noise.

Example 2.34. Let $(T, \mathscr{B}, \mu) = (\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), m)$ where m is the Lebesgue measure. Then W is the white noise in $\mathbb{R}_+$ and $B_t := W([0,t])$ is the one dimensional Brownian motion. The details are similar to Example 2.5.

Fix $n \in \mathbb{N}$ and let $\mathscr{B}_b(T) := \{A \in \mathscr{B}(T) : \mu(A) < \infty\}$. Further let E_n be the set of elementary functions of the form

$$f(t_1, \cdots, t_n) = \sum_{i_1, \cdots, i_n = 1}^{n} a_{i_1, \cdots, i_n} 1_{A_{i_1} \times \cdots \times A_{i_n}}(t_1, \cdots, t_n) \tag{2.8}$$

where $A_1, \cdots, A_n \in \mathscr{B}_b(T)$ are pairwise disjoint and the coefficients $a_{i_1, \cdots, i_n} = 0$ if $i_j = i_k$ for any $j \neq k$.

Proposition 2.35. The set of elementary functions E_n is dense in $L^2(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$.

Definition 2.36. For an elementary function of the form (2.8), *the multiple stochastic integral* is defined as follows:

$$I_n(f) = \sum_{i_1,\cdots,i_n}^{m} a_{i_1,\cdots,i_n} W(A_{i_1}) \cdots \times W(A_{i_n}) \tag{2.9}$$

Let $L^2_s(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$ denote the space of symmetric square integrable functions. If $f : T^n \to \mathbb{R}$, define its symmetrization by

$$\tilde{f}(t_1, \cdots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) \tag{2.10}$$

where sum runs over all permutations σ of $\{1, \cdots, n\}$. Observe $\|\tilde{f}\|_{L^2(T^n,\mathscr{B}(T^n),\mu^{\otimes n})} \leq \|f\|_{L^2(T^n,\mathscr{B}(T^n),\mu^{\otimes n}))}$.

Proposition 2.37. (i) The definition (2.9) doesn't depend on the particular representation of the function f.

(ii) Let $\tilde{f}$ denote the symmetrization of $f \in E_n$ as defined in (2.10). Then

$$I_n(f) = I_n(\tilde{f}).$$

Lemma 2.38. For all $n, m \in \mathbb{N}$ and $f \in E_n$ and $g \in E_m$, we have

$$\mathrm{E}\left[I_n(f) I_m(g)\right] = \begin{cases} 0 & \text{if } n \neq m \\ n! \langle \tilde{f}, \tilde{g} \rangle_{L^2(T^n,\mathscr{B}(T^n),\mu^{\otimes n})} & \text{if } n = m. \end{cases}$$

Proposition 2.39. The linear operator $I_n : E_n \to L^2(\Omega, \mathfrak{F}, \mathrm{P})$ can be extended to a continuous linear operator from $L^2(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$ to $L^2(\Omega, \mathfrak{F}, \mathrm{P})$. Moreover, for all $f, g \in L^2(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$,

$I_n(f) = I_n(\tilde{f})$ and

$$\mathrm{E}\left[I_n(f)I_m(g)\right] = \begin{cases} 0 & \text{if } n \neq m \\ n!\langle \tilde{f}, \tilde{g}\rangle_{L^2(T^n,\mathscr{B}(T^n),\mu^{\otimes n})} & \text{if } n = m. \end{cases}$$

still holds.

Definition 2.40. Let $f \in L^2(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$ and $g \in L^2(T^m, \mathscr{B}(T^m), \mu^{\otimes m}))$. For any $r = 0, \cdots, m \wedge n$, define *the contraction of* f and g of order r to be the element of $L^2(T^{n+m-2r}, \mathscr{B}(T^{n+m-2r}), \mu^{\otimes n+m-2r})$ as

$$\begin{aligned}
&(f \otimes_r g)(t_1, \cdots, t_{n-r}, s_1, \cdots s_{m-r}) \\
&= \int_{T^r} f(t_1, \cdots, t_{n-r}, x) g(s_1, \cdots s_{m-r}, x) \mu^r(dx).
\end{aligned}$$

Denote the symmetrization of $f \otimes_r g$ by $f \tilde{\otimes}_r g$.

Proposition 2.41. Let $f \in L^2_s(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$ and $g \in L^2_s(T^m, \mathscr{B}(T^m), \mu^{\otimes m})$ for some $m, n \in \mathbb{N}_0$. Then,

$$I_n(f)I_m(g) = \sum_{r=0}^{n\wedge m} r! \binom{m}{r}\binom{n}{r} I_{n+m-2r}(f \otimes_r g).$$

Proposition 2.42. For any $g \in L^2(T, \mathscr{B}(T), \mu)$, we have

$$I_n(g^{\otimes n}) = n! \|g\|^n_{L^2(T,\mathscr{B}(T),\mu)} H_n\left(\frac{W(g)}{\|g\|_{L^2(T,\mathscr{B}(T),\mu)}}\right),$$

where $g^{\otimes n}(t_1, \cdots, t_n) = g(t_1)\cdots g(t_n)$. In particular, if $\|g\|_{L^2(T,\mathscr{B}(T),\mu)} = 1$, then

$$I_n(g^{\otimes n}) = n! H_n\left(W(g)\right).$$

As a consequence of Proposition 2.42 and Theorem 2.13, we deduce following version of the

Wiener chaos expansion.

Theorem 2.43. Every $F \in L^2(\Omega, \mathfrak{F}, \mathrm{P})$ can be uniquely expanded into a sum of stochastic integrals as follows:

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_0 = \mathrm{E}[F]$, and I_0 denotes the identity map on constants and $f_n \in L^2(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$. Further we can assume that the functions $f_n \in L_s^2(T^n, \mathscr{B}(T^n), \mu^{\otimes n})$ and in this case are uniquely determined by F.

Using chaos expansion we can easily compute the derivative as follows:

Proposition 2.44. Let $F \in \mathbb{D}^{1,2}$ be a random variable with chaos expansion given in Theorem 2.43 where f_n's are symmetric . Then,

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)).$$

Proof. Let $F = I_n(f_n)$ for a particular $n \in \mathbb{N}_0$ where f_n is symmetric and of the form (2.8) and assume $t \in A_{i_j}$ for some $i_j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}
D_t F &= D_t \left(\sum_{i_1, \cdots, i_n = 1}^{n} a_{i_1, \cdots, i_n} W(A_{i_1}) \cdots W(A_{i_n}) \right) \\
&= \sum_{j=1}^{n} \sum_{i_1, \cdots, i_n = 1}^{n} a_{i_1, \cdots, i_n} W(A_{i_1}) \cdots \mathbf{1}_{A_{i_j}}(t) \cdots W(A_{i_n}) \\
&= n I_{n-1}(f_n(\cdot, t)).
\end{aligned}$$

The general case then follows. $\square$

Proposition 2.45. Let $F \in L^2(\Omega, \mathfrak{F}, \mathrm{P})$ be a random variable with chaos expansion given in Theorem 2.43 and $A \in \mathscr{B}(T)$. Then,

$$\mathrm{E}\left[F \middle| \mathfrak{F}_t\right] = \sum_{n=0}^{\infty} I_n(f_n(\cdot)\mathbf{1}_{[0,t]}^{\otimes n}).$$

Proof. Let $F = I_n(f_n)$ for a particular $n \in \mathbb{N}_0$ where f_n of the form $\mathbf{1}_{A_1 \times \cdots \times A_n}$ where $A_1, \cdots A_n \in \mathscr{B}_b(T)$ mutually disjoint. Then

$$\begin{aligned}\mathrm{E}\left[F \middle| \mathfrak{F}_t\right] &= \mathrm{E}\left[W(A_1)\cdots W(A_n) \middle| \mathfrak{F}_t\right] = \mathrm{E}\left[\prod_{i=1}^{n} W(A_i \cap [0,t]) + W(A_i \cap [0,t]^c) \middle| \mathfrak{F}_t\right] \\ &= I_n(\mathbf{1}_{(A_1\cap[0,t]\times\cdots\times(A_n\cap[0,t])}).\end{aligned}$$

The general case then follows. □

An element $V(t) \in L^2(T \times \Omega, \mathscr{B}(T) \otimes \mathfrak{F}, m \otimes \mathrm{P})$ has a Wiener chaos decomposition of the form

$$V(t) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)), \tag{2.11}$$

where for each $n \in \mathbb{N}$, $f_n \in L^2(T^{n+1})$ which is symmetric in the first n-components. The next result shows how divergence operator applies to Wiener chaos decomposition.

Proposition 2.46. Let $V(t) \in L^2(T \times \Omega, \mathscr{B}(T) \otimes \mathfrak{F}, m \otimes \mathrm{P})$ be given as in (2.11). Then $V \in \mathrm{Dom}\delta$ if and only if the series

$$\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

where

$$\tilde{f}_n(t_1,t_2,\cdots,t_n,t) := \frac{1}{n+1}\left(f_n(t_1,\cdots,t_n,t) + \sum_{i=1}^{n} f_n(t_1,\cdots,t_{i-1},t,t_{i+1}\cdots,t_n,t_i)\right)$$

converges in $L^2(\Omega,\mathfrak{F},\mathrm{P})$. Moreover,

$$\delta(V)=\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n).$$

2.5 Itô integral and Malliavin calculus

We will focus on Example 2.34 where $\mathfrak{H}=L^2(\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+),m)$ throughout this section. Following Remark 2.31 we will use the identification $L^2(\Omega,\mathfrak{F},\mathrm{P};L^2(\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+),m))\cong L^2(\mathbb{R}_+\times\Omega,\mathscr{B}(\mathbb{R}_+)\otimes\mathfrak{F},m\otimes\mathrm{P})$. Thus the Malliavin derivative is a stochastic process $(D_tF)_{t\in\mathbb{R}_+}$. Let $(\mathfrak{F}_t)_{t\in\mathbb{R}_+}$ be the filtration such that

$$\mathfrak{F}_t=\mathfrak{F}_t^0\vee\mathscr{N},\ \ \mathfrak{F}_t^0:=\sigma(B_s,0\leq s\leq t) \tag{2.12}$$

where $\mathscr{N}$ is the σ-field generated by P-null sets. We say a process $\{V_t\}_{t\in\mathbb{R}_+}$ is adapted if V_t is $\mathfrak{F}_t$-measurable for all $t\in\mathbb{R}_+$. Let $L_a^2(\mathbb{R}_+\times\Omega,\mathscr{B}(\mathbb{R}_+)\otimes\mathfrak{F},m\otimes\mathrm{P})$ be the set of square integrable and adapted processes. Further, let $\mathscr{E}(\mathbb{R}_+)$ denote the set of all finite linear combinations of elementary adapted processes of the form

$$V(s)=F\mathbf{1}_{[a,b)}(s) \tag{2.13}$$

where $0<a<b<\infty$, $F\in L^2(\Omega,\mathfrak{F},\mathrm{P})$, $\mathfrak{F}_a$-measurable. Recall that for an elementary adapted process of the type (2.13), the Itô integral is given by

$$\int_{\mathbb{R}_+} V(s)dB_s=F(B_b-B_a)=FW(\mathbf{1}_{[a,b)}).$$

Theorem 2.47. The space $L^2_a(\Omega,\mathfrak{F},\mathrm{P};L^2(\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+),m))$ is included in the domain of δ, moreover

$$\delta(V)=\int_{\mathbb{R}_+}V(s)dB_s$$

for any $V\in L^2_a(\Omega,\mathfrak{F},\mathrm{P};L^2(\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+),m))$.

Proof. Let $V(s)=F\mathbf{1}_{[a,b)}(s)$ where $F\in\mathscr{S}$. Then for any $G\in\mathscr{S}$, we have

$$\mathrm{E}\left[\langle V,DG\rangle_{\mathfrak{H}}\right]=\mathrm{E}\left[F\langle\mathbf{1}_{[a,b)},DG\rangle_{\mathfrak{H}}\right]=\mathrm{E}\left[FGW(\mathbf{1}_{[a,b)})-G\langle 1_{[a,b)},DF\rangle_{\mathfrak{H}}\right]\tag{2.14}$$

where we used (2.5) in the last equality. Note that since $F\in\mathscr{S}$ is $\mathscr{F}_a$-measurable, and $F=f(W(h_1),\cdots,W(h_n))$ for some smooth f and $h_i\in L^2(\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+),m)$ such that supp $h_i\subset[0,a]$. This implies, in particular, $\langle\mathbf{1}_{[a,b)},h_i\rangle=0$ for all $i=1,\cdots m$ and $\langle\mathbf{1}_{[a,b)},DF\rangle=0$. So, the above identity becomes

$$\mathrm{E}\left[\langle V,DG\rangle_{\mathfrak{H}}\right]=\mathrm{E}\left[FGW(1_{[a,b)}\right]$$

which can be rewritten using (2.14) as

$$\mathrm{E}\left[\langle V,DG\rangle_{\mathfrak{H}}\right]=\mathrm{E}\left[F\int_{\mathbb{R}_+}V(s)dB_s\right].$$

The proof can then be completed by an approximation argument. □

The following theorem includes Clark-Ocone Formula and Poincare inequality.

Theorem 2.48. For every $F\in\mathbb{D}^{1,2}$,

$$F=\mathrm{E}[F]+\int_{\mathbb{R}_+}\mathrm{E}\left[D_sF|\mathfrak{F}_s\right]dB_s,\ \text{ a.s.}$$

Consequently, we have the Poincare inequality:

$$\operatorname{Var}[F] \le \operatorname{E}\left[\|DF\|^2_{L^2(\mathbb{R}_+,\mathscr{B}(\mathbb{R}_+),m)}\right]$$

Proof. By martingale representation theorem, there is a unique process $V \in L^2_a(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathfrak{F}, m \otimes \mathrm{P})$ such that

$$F = \operatorname{E}[F] + \int_{\mathbb{R}_+} U(s) dB_s. \tag{2.15}$$

Let $U \in L^2_a(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathfrak{F}, m \otimes \mathrm{P})$. On one hand, using the isometry property of Itô integral, we see

$$\operatorname{E}[\delta(U)F] = \int_{\mathbb{R}_+} \operatorname{E}[U(s)V(s)]\, ds.$$

On the other hand, using integration by parts (2.7), we get

$$\operatorname{E}[\delta(U)F] = \operatorname{E}\left[\int_0^\infty U(t) D_t F dt\right] = \int_0^\infty \operatorname{E}\left[U(s)\operatorname{E}\left[D_s F \middle| \mathfrak{F}_s\right]\right] ds$$

where we used the fact that U is adapted to the filtration $\{\mathfrak{F}_s\}_{s\in\mathbb{R}_+}$. The above findings together implies $V(s) = \operatorname{E}\left[D_s F \middle| \mathfrak{F}_s\right]$. ☐

Remark 2.49. Another way to prove Clark-Ocone formula is using chaos expansion in Theorem 2.43 together with Proposition 2.44, Proposition 2.45 and Proposition 2.46. See [42, Proposition 1.3.14] for details.

2.6 Existence of density

The following density formula under general assumptions on the random variable has been proved in [9, Proposition 1].

Proposition 2.50. Let $F \in \mathbb{D}^{1,1}$ and $V \in L^1(\Omega, \mathfrak{F}, \mathrm{P}; \mathfrak{H})$ be such that $D_V F \neq 0$ a.s. Assume that $V/D_V F \in \operatorname{Dom} \delta$. Then the law of F has a continuous and bounded density given by

$$f_F(x) = \mathrm{E}\left[\mathbf{1}_{[F>x]}\delta\left(\frac{V}{D_V F}\right)\right].$$

Remark 2.51. Using Lemma 2.30, in the context of Proposition 2.50, the following constitute sufficient conditions for $V/D_V F \in \operatorname{Dom} \delta$, for some p, p' with $1/p + 1/p' = 1$ (see [9, Lemma 3]):

(i) $(D_V F)^{-1} \in \mathbb{D}^{1,p'}$.

(ii) $V \in \mathbb{D}^{1,p}(\mathfrak{H})$.

In view of [9, Lemma 4], a sufficient condition for (i) is $(D_V F)^{-1} \in L^{p'}(\Omega, \mathfrak{F}, \mathrm{P})$ and

$$(D_V F)^{-2}\left[\|D^2 F\|_{\mathfrak{H}\otimes\mathfrak{H}}\|V\|_{\mathfrak{H}} + \|DV\|_{\mathfrak{H}\otimes\mathfrak{H}}\|DF\|_{\mathfrak{H}}\right] \in L^{p'}(\Omega, \mathfrak{F}, \mathrm{P}).$$

Therefore, assuming that $F \in \mathbb{D}^{2,p}$ and $(D_V F)^{-1} \in L^q(\Omega, \mathfrak{F}, \mathrm{P})$, then condition (i) holds if $2/q + 3/p = 1$ for some $p > 3$ and $q > 2$. In particular, we can take $q = 4$ and $p = 6$.

Proof of Proposition 2.50. Let $\psi \in C_c^\infty(\mathbb{R}; \mathbb{R}_+)$ and define $\varphi(y) = \int_{-\infty}^y \psi(z)dz$ for $y \in \mathbb{R}$. Then by Proposition 2.24, we have $\varphi(F) \in \mathbb{D}^{1,1}$ and

$$\langle D(\varphi(F)), V\rangle_{\mathfrak{H}} = \psi(F)\langle DF, V\rangle_{\mathfrak{H}}.$$

Using $D_V F \neq 0$ a.s., we obtain

$$\mathrm{E}[\psi(F)] = \mathrm{E}\left[\left\langle D(\varphi(F)), \frac{V}{\langle DF, V\rangle}\right\rangle_{\mathfrak{H}}\right].$$

Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathscr{S}$ be a sequence of random variables converging to F in $\mathbb{D}^{1,1}$. Then, using the definition of the divergence operator δ, we get

$$\begin{aligned}
\mathrm{E}\left[\left\langle D(\varphi(F)), \frac{V}{\langle DF, V\rangle_{\mathfrak{H}}}\right\rangle_{\mathfrak{H}}\right] &= \lim_{n\to\infty} \mathrm{E}\left[\left\langle D(\varphi(F_n)), \frac{V}{\langle DF, V\rangle_{\mathfrak{H}}}\right\rangle_{\mathfrak{H}}\right] \\
&= \lim_{n\to\infty} \mathrm{E}\left[\varphi(F_n)), \delta\left(\frac{V}{\langle DF, V\rangle_{\mathfrak{H}}}\right)\right] \\
&= \mathrm{E}\left[\varphi(F))\delta\left(\frac{V}{\langle DF, V\rangle_{\mathfrak{H}}}\right)\right].
\end{aligned}$$

Hence, we get

$$\mathrm{E}\left[\psi(F)\right] = \mathrm{E}\left[\varphi(F)\delta\left(\frac{V}{\langle DF, V\rangle_{\mathfrak{H}}}\right)\right]. \tag{2.16}$$

By an approximation argument, (2.16) holds for the function $\psi(y) = \mathbf{1}_{[a,b]}(y)$ and as a consequence, applying Fubini's theorem, we obtain

$$\begin{aligned}
\mathrm{P}\left(a \leq F \leq b\right) &= \mathrm{E}\left[\left(\int_{-\infty}^{F} \mathbf{1}_{[a,b]}(x)dx\right)\delta\left(\frac{V}{\langle DF, V\rangle_{\mathfrak{H}}}\right)\right] \\
&= \int_a^b \mathrm{E}\left[\mathbf{1}_{[F>x]}\delta\left(\frac{V}{\langle DF, V\rangle_{\mathfrak{H}}}\right)\right] dx
\end{aligned}$$

which concludes the proof of the claim. $\square$

Chapter 3

Malliavin-Stein method

Stein's method [54] was established in 1970s to provide quantitative results to estimate how far a random variable is from being normal. Before Stein [54] first used the method, a Fourier transform approach was used (characteristic functions) to show the convergence to a normal random variable in distribution. Although this method is a strong tool to establish convergence in distribution, it lacks to provide the estimates on the error term in general. In 2005, Nualart and Peccati [44] formulated a new central limit theorem on a fixed Wiener chaos, which is called *the fourth moment theorem*. Later in 2009, Nourdin and Peccati [37], when considering the rate of convergence for the fourth moment theorem, explored an interplay between Malliavin calculus and Stein's method leading to quantitative estimates. This chapter is based on the books Chen, Goldstein, and Shao [17], Nourdin and Peccati [38], Nualart [42], as well as the surveys Nourdin [36], Nourdin and Peccati [37], Ross [51]. At the end of this chapter we will recall with its proofs particular Malliavin-Stein bounds one for Wasserstein distane and other for the uniform distance between densities. These estimates are based on the results in Kuzgun and Nualart [30] and Hu, Lu, and Nualart [24], Kuzgun and Nualart [31].

3.1 Stein's method

The following is an important characterization of the normal distribution.

Lemma 3.1 (Stein's Lemma)**.** If Z has the standard normal distribution, then

$$\mathrm{E}\left[\varphi'(Z) - Z\varphi(Z)\right] = 0$$

for all absolutely continuous function φ with $\varphi' \in L^1(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$. On the other hand, if F is a random variable such that

$$\mathrm{E}\left[\varphi'(F) - F\varphi(F))\right] = 0$$

for all absolutely continuous function φ with $\varphi' \in L^1(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ with $\mathrm{E}[F\varphi(F)] < \infty$, then F has the standard normal distribution.

This characterization of the normal distribution motivates the Stein equation (3.2).

Proposition 3.2. For $\varphi \in L^1(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$, the function

$$f_\varphi(x) := e^{x^2/2} \int_x^\infty (\varphi(y) - \mathrm{E}[\varphi(Z)])\, e^{-y^2/2} dy. \tag{3.1}$$

is the unique solution to the Stein's equation

$$f(x) - xf'(x) = \varphi(x) - \mathrm{E}[\varphi(Z)] \tag{3.2}$$

satisfying the growth condition

$$\lim_{x \to \pm\infty} e^{-x^2/2} f(x) = 0.$$

The following lemma presents some properties of the solution f_φ to Stein's equation for particular choices of φ.

Lemma 3.3.

(i) Let $\varphi(y) = \mathbf{1}_{(-\infty,x)}(y)$ for some $x \in \mathbb{R}$. Then,

$$\|f_\varphi\|_{L^\infty} \leq \frac{\sqrt{2\pi}}{4}, \quad \|f'_\varphi\|_{L^\infty} \leq 1.$$

(ii) Let $\varphi : \mathbb{R} \to [0,1]$ be Borel measurable. Then,

$$\|f_\varphi\|_{L^\infty} \le \sqrt{\frac{\pi}{2}}, \quad \|f'_\varphi\|_{L^\infty} \le 2.$$

(iii) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, then

$$\|f_\varphi\|_{L^\infty} \le 2\|\varphi'\|_{L^\infty}, \quad \|f'_\varphi\|_{L^\infty} \le 4\|\varphi'\|_{L^\infty}, \quad \|f''_\varphi\|_{L^\infty} \le 2\|\varphi'\|_{L^\infty}.$$

Heuristically, Stein's method aims to use the characterization in Lemma 3.1 to estimate how far a random variable is from being normally distributed. Before we state the results in this respect, let us first introduce how we measure the distance between two random variables.

Definition 3.4. For two real random variables F, G on a probability space $(\Omega, \mathfrak{F}, \mathrm{P})$, define

$$d_{\mathscr{C}}(F,G) := \sup_{\varphi \in \mathscr{C}} |\mathrm{E}[\varphi(F)] - \mathrm{E}[\varphi(G)]|$$

where $\mathscr{C}$ is an appropriate class of test functions.

We will mainly be interested in the following cases:

Definition 3.5.

(i) If we take $\mathscr{C} := \{\mathbf{1}_{(-\infty,x]}; x \in \mathbb{R}\}$, then we obtain the *Kolmogorov's distance*:

$$\mathrm{d_{Kol}}(F,G) := \sup_{x \in \mathbb{R}} |\mathrm{P}(F \le x) - \mathrm{P}(G \le x)|$$

(ii) If we take $\mathscr{C} := \{\mathbf{1}_B(\cdot) : B \in \mathscr{B}(\mathbb{R})\}$, we get the *total variation distance*:

$$\mathrm{d_{TV}}(F,G) := \sup_{B \in \mathbb{B}(\mathbb{R})} |\mathrm{P}(F \in B) - \mathrm{P}(G \in B)|$$

(iii) If we take $\mathscr{C} := \{\varphi : \mathbb{R} \to \mathbb{R} : |\varphi(x) - \varphi(y)| \le |x - y| \text{ for all } x, y \in \mathbb{R}\} =: \mathrm{Lip}(1)$, we get the *Wasserstein's distance*:

$$\mathrm{d_W}(F,G) := \sup_{\varphi \in \mathrm{Lip}(1)} |\mathrm{E}[\varphi(F)] - \mathrm{E}[\varphi(G)]|\,.$$

The following proposition is the central result in Stein's method which characterizes the distances in term of the solution to the Stein's equation (3.2).

Proposition 3.6. Let F be an integrable random variable and Z has a normal distribution, then

$$\mathrm{d}_{\mathscr{C}}(F,Z) = \sup_{\varphi \in \mathscr{C}} \mathrm{E}\left[f'_\varphi(F) - F f_\varphi(F)\right]$$

for any class of functions $\mathscr{C}$ introduced in Definition 3.5 where f_φ given in (3.1) is the solution to the Stein equation (3.2).

Using Lemma 3.3 and Proposition 3.6 we obtain following corollaries which will be used when we introduce the Malliavin-Stein method in the next section.

Corollary 3.7. Let F be an integrable random variable and Z has a normal distribution, then

$$\mathrm{d_{Kol}}(F,Z) \le \sup_{f \in \mathscr{F}_{\mathrm{Kol}}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}[F f(F)]\right|$$

where $\mathscr{F}_{\mathrm{Kol}}$ is the class of piecewise continuously differentiable functions where $\|f\|_{L^\infty} \le \sqrt{2\pi}/4$ and $\|f'\|_{L^\infty} \le 1$.

Corollary 3.8. Let F be an integrable random variable and Z has a normal distribution, then

$$\mathrm{d_{TV}}(F,Z) \le \sup_{f \in \mathscr{F}_{\mathrm{TV}}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}[F f(F)]\right|$$

where $\mathscr{F}_{\mathrm{TV}}$ is the class of absolutely continuous functions $\|f\|_{L^\infty} \le \sqrt{\pi/2}$ whose derivative has a version such that $\|f'\|_{L^\infty} \le 2\}$.

Corollary 3.9. Let F be an integrable random variable and Z has a normal distribution, then

$$\mathrm{d_W}(F,Z) \leq \sup_{f \in \mathscr{F}_\mathrm{W}} \left| \mathrm{E}\left[f'(F)\right] - \mathrm{E}\left[F f(F)\right] \right|$$

where $\mathscr{F}_\mathrm{W}$ is the class of twice differentiable functions such that $\|f\|_{L^\infty} \leq 2$, $\|f'\|_{L^\infty} \leq \sqrt{\pi/2}$, $\|f''\|_{L^\infty} \leq 2\}$.

3.2 Stein's method combined with Malliavin calculus

Now, we are ready to combine Malliavin calculus with Stein's method to give estimates for the distances introduced in the previous section. The theorems below are stated for the random variables of the form $F = \delta(V)$ for some $V \in \mathrm{Dom}(\delta)$ which suits for our purposes in the later chapters. For a broader treatment and the proofs, see [38]. These estimates are first obtained in [37]. Throughout this section, we fix an isonormal Gaussian process W over $\mathfrak{H}$ on a probability space $(\Omega, \mathfrak{F}, \mathrm{P})$.

Theorem 3.10. Suppose that $F \in \mathbb{D}^{1,2}$ satisfies $F = \delta(V)$ where V belongs to the $\mathrm{Dom}(\delta)$ and $\mathrm{E}\left[F^2\right] = 1$. Let $\mathscr{C}$ be the one of the classes of functions defined in Definition 3.5. Then,

$$\mathrm{d}_\mathscr{C}(F,Z) \leq \left(\sup_{\varphi \in \mathscr{C}} \|f'_\varphi\|_{L^\infty} \right) \sqrt{\mathrm{Var}\left[\langle DF, V \rangle_\mathfrak{H}\right]},$$

where Z is a standard normal random variable. In particular,

$$\mathrm{d_{Kol}}(F,Z) \leq \sqrt{\mathrm{Var}\left[\langle DF, V \rangle_\mathfrak{H}\right]}, \tag{3.3}$$

$$\mathrm{d_{TV}}(F,Z) \leq 2\sqrt{\mathrm{Var}\left[\langle DF, V \rangle_\mathfrak{H}\right]}, \tag{3.4}$$

$$\mathrm{d_W}(F,Z) \leq \sqrt{\pi/2}\sqrt{\mathrm{Var}\left[\langle DF, V \rangle_\mathfrak{H}\right]}. \tag{3.5}$$

Proof. Let us prove this for the total variation distance. The others follow the same steps. By (2.4), we see $\mathrm{E}[\delta(V)f(F)] = \mathrm{E}[\langle D(f(F)),V\rangle_{\mathfrak{H}}]$. As a consequence, using Corollary 3.8, we can write

$$\begin{aligned}
\mathrm{d}_{\mathrm{TV}}(F,Z) &\leq \sup_{f\in\mathscr{F}_{\mathrm{TV}}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}[Ff(F)]\right| \\
&= \sup_{f\in\mathscr{F}_{\mathrm{TV}}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}[\delta(V)f(F)]\right| \\
&= \sup_{f\in\mathscr{F}_{\mathrm{TV}}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}[\langle D(f(F)),V\rangle_{\mathfrak{H}}]\right| \\
&= \sup_{f\in\mathscr{F}_{\mathrm{TV}}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}\left[f'(F)\langle DF,V\rangle_{\mathfrak{H}}\right]\right| \\
&\leq 2\mathrm{E}[|1-\langle DF,V\rangle_{\mathfrak{H}}|],
\end{aligned}$$

where we used Proposition 2.24 and $f\in\mathscr{F}_{\mathrm{TV}}$ satisfies $\|f'\|_{L^\infty}\leq 2$. Since $1=\mathrm{E}\left[F^2\right]=\mathrm{E}[F\delta(V)]=\mathrm{E}[\langle DF,V\rangle_{\mathfrak{H}}]$, using Cauchy-Schwarz inequality, we get

$$\mathrm{E}[|1-\langle DF,V\rangle_{\mathfrak{H}}|] \leq \sqrt{\mathrm{E}\left[|\mathrm{E}[\langle DF,V\rangle_{\mathfrak{H}}]-\langle DF,V\rangle_{\mathfrak{H}}|^2\right]} = \sqrt{\mathrm{Var}[\langle DF,V\rangle_{\mathfrak{H}}]},$$

which concludes our proof. □

An iterative application of the Malliavin-Stein approach leads to the following result, which requires the random variable $F=\delta(V)$ to be three times differentiable (see [46, Proposition 3.2.]).

Proposition 3.11. Assume that $V\in\mathrm{Dom}\,\delta$, $F=\delta(V)\in\mathbb{D}^{3,2}$ and $\mathrm{E}\left[F^2\right]=1$. Then,

$$\mathrm{d}_{\mathrm{TV}}(F,Z)\leq(8+\sqrt{32\pi})\mathrm{Var}[\langle DF,V\rangle_{\mathfrak{H}}]+\sqrt{2\pi}\left|\mathrm{E}\left[F^3\right]\right|+\sqrt{32\pi}\mathrm{E}\left[|D_V^2F|^2\right]+4\pi\mathrm{E}\left[|D_V^3F|\right],$$

where we have used the notation $D_VF=\langle V,DF\rangle_{\mathfrak{H}}$ and $D_V^{i+1}F=\langle V,D(D_V^iF)\rangle_{\mathfrak{H}}$ for $i\geq 1$.

In the next proposition we present another estimate for the Wasserstein's distance between a random variable F where $F=\delta^2(V)$ and a normal random variable obtained using iterative application of Malliavin-Stein method. This is proved in [30].

Proposition 3.12. Assume that $V \in \operatorname{Dom}\left(\delta^2\right)$, $F = \delta^2(V) \in \mathbb{D}^{2,2}$ and $\mathrm{E}\left[F^2\right] = 1$. Then,

$$\mathrm{d_W}(F,Z) \leq \sqrt{\pi/2}\sqrt{\operatorname{Var}\left[\langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right]} + 2\mathrm{E}\left[\left|\langle DF \otimes DF, V\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right].$$

Proof. By iterating (2.4), we see $\mathrm{E}\left[\delta^2(V)f(F)\right] = \mathrm{E}\left[\langle D^2(f(F)), V\rangle_{\mathfrak{H}^{\otimes 2}}\right]$. As a consequence, using Corollary 3.9, we can write

$$\begin{aligned}
\mathrm{d_W}(F,Z) &\leq \sup_{f\in\mathscr{F}_\mathrm{W}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}\left[Ff(F)\right]\right| \\
&= \sup_{f\in\mathscr{F}_\mathrm{W}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}\left[\delta^2(V)f(F)\right]\right| \\
&= \sup_{f\in\mathscr{F}_\mathrm{W}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}\left[\langle D^2(f(F)), V\rangle_{\mathfrak{H}^{\otimes 2}}\right]\right| \\
&= \sup_{f\in\mathscr{F}_\mathrm{W}} \left|\mathrm{E}\left[f'(F)\right] - \mathrm{E}\left[f'(F)\langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right] - \mathrm{E}\left[f''(F)\langle DF\otimes DF, V\rangle_{\mathfrak{H}^{\otimes 2}}\right]\right| \\
&\leq \sqrt{\pi/2}\,\mathrm{E}\left[\left|1 - \langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right] + 2\mathrm{E}\left[\left|\langle DF\otimes DF, V\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right],
\end{aligned}$$

where we used Proposition 2.24 and $f \in \mathscr{F}_\mathrm{W}$ satisfies $\|f'\|_{L^\infty} \leq \sqrt{\pi/2}$, $\|f''\|_{L^\infty} \leq 2\}$. Since $1 = \mathrm{E}\left[F^2\right] = \mathrm{E}\left[F\delta^2(V)\right] = \mathrm{E}\left[\langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right]$, using Cauchy-Schwarz inequality, we get

$$\mathrm{E}\left[\left|1 - \langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right] \leq \sqrt{\mathrm{E}\left[\left|\mathrm{E}\left[\langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right] - \langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right|^2\right]} = \sqrt{\operatorname{Var}\left[\langle D^2F, V\rangle_{\mathfrak{H}^{\otimes 2}}\right]},$$

which concludes our proof. □

Now, we will use Malliavin-Stein method to obtain a bound for the uniform distance between the density of of a random variable and the density of the normal distribution. In order to obtain such estimate, we will use Proposition 2.50. Variations of this result are obtained in [24]. The proof here is given in [31].

Theorem 3.13. Assume that $V \in \mathbb{D}^{1,6}(\mathfrak{H})$ and $F = \delta(V) \in \mathbb{D}^{2,6}$ with $\mathrm{E}[F] = 0$, $\mathrm{E}\left[F^2\right] = 1$ and $(D_VF)^{-1} \in L^4(\Omega, \mathfrak{F}, \mathrm{P})$. Then, $V/D_VF \in \operatorname{Dom}\delta$, F admits a density $f_F(x)$ and the following inequality holds true

$$\sup_{x\in\mathbb{R}}|f_F(x)-\phi(x)| \leq \left(\|F\|_4 \left\|(D_V F)^{-1}\right\|_4 + 2\right)\|1-D_V F\|_2 + \left\|(D_V F)^{-1}\right\|_4^2 \|D_V(D_V F)\|_2, \tag{3.6}$$

where $\phi(x)$ is the density of the normal distribution.

Proof. First, note that, by Proposition 2.50, the Remark 2.51 F admits a density $f_F(x) = \mathrm{E}\left[\mathbf{1}_{[F>x]}\delta(\bar{V})\right]$, where $\bar{V} = V/D_V F$. As a consequence, we can write

$$\sup_{x\in\mathbb{R}}|f_F(x)-\phi(x)| = \sup_{x\in\mathbb{R}}\left|\mathrm{E}\left[\mathbf{1}_{[F>x]}\delta(\bar{V})\right] - \mathrm{E}\left[\mathbf{1}_{[Z>x]}Z\right]\right|, \tag{3.7}$$

where Z denotes a standard random variable. We have

$$\delta(\bar{v}) = \delta\left(\frac{V}{D_V F}\right) = \frac{F}{D_V F} - D_V\left(\frac{1}{D_V F}\right) = \frac{F}{D_V F} + \frac{D_V(D_V F)}{(D_V F)^2}. \tag{3.8}$$

Indeed, the second equality follows from Lemma 2.30 together with $F = \delta(V)$, and the third one follows from the chain rule. Then, substituting (3.8) into (3.7), yields

$$\begin{aligned}\Phi_x &:= \left|\mathrm{E}\left[\mathbf{1}_{[F>x]}\delta(\bar{V})\right] - \mathrm{E}\left[\mathbf{1}_{[Z>x]}Z\right]\right| \\ &= \left|\mathrm{E}\left[\frac{\mathbf{1}_{[F>x]}F}{D_V F}\right] - \mathrm{E}\left[\frac{\mathbf{1}_{[F>x]}D_V(D_V F)}{(D_V F)^2}\right] - E\left[\mathbf{1}_{\{Z>x\}}Z\right]\right|.\end{aligned} \tag{3.9}$$

Adding and subtracting $\mathrm{E}\left[\mathbf{1}_{[F>x]}F\right]$ in (3.9), we get

$$\Phi_x \leq \mathrm{E}\left[\left|\frac{(1-D_V F)F}{D_V F}\right|\right] + \mathrm{E}\left[\frac{|D_V(D_V F)|}{(D_V F)^2}\right] + \left|\mathrm{E}\left[F\mathbf{1}_{[F>x]} - Z\mathbf{1}_{[Z>x]}\right]\right|. \tag{3.10}$$

Applying Hölder's inequlity to the first term, we obtain

$$\mathrm{E}\left[\left|\frac{(1-D_V F)F}{D_V F}\right|\right] \leq \|F\|_4 \left\|(D_V F)^{-1}\right\|_4 \|1-D_V F\|_2. \tag{3.11}$$

Meanwhile, applying Hölder's inequality to the second term, we get

$$\mathrm{E}\left[\frac{|D_V(D_VF)|}{(D_VF)^2}\right] \leq \left\|(D_VF)^{-1}\right\|_4^2 \|D_V(D_VF)\|_2. \tag{3.12}$$

Finally, applying Stein's method Theorem 3.10 with $\varphi(y) = y\mathbf{1}_{[y>x]}$ which is a Lip(1) function, we obtain

$$\left|\mathrm{E}\left[F\mathbf{1}_{[F>x]} - Z\mathbf{1}_{[Z>x]}\right]\right| \leq \sqrt{\pi/2}\sqrt{\mathrm{Var}\left[\langle DF, V\rangle_{\mathfrak{H}}\right]}. \tag{3.13}$$

Then, substituting (3.11), (3.12) into (3.10) yields the desired estimate.

□

Chapter 4

Walsh stochastic integral

Walsh introduced multi-parameter stochastic integration which is an extension of Itô's calculus in the seminal paper [56]. In this chapter, we will first give a brief sketch of this integration theory in the context of a spatially homogeneous noise and recall the facts that the main results of Itô's theory extends to Walsh integration. Afterwards, we present how this theory connects with Malliavin calculus. The first two section of this chapter is mainly based on lecture notes Perkowski [50]. We also utilized the lecture notes Balan [2], the papers Dalang [20], Walsh [56], the books Khoshnevisan [28], Dalang, Khoshnevisan, Mueller, Nualart, and Xiao [21], and the theses Chen [10], Conus [18]. The last section is based on a recent paper Chen, Khoshnevisan, Nualart, and Pu [16].

4.1 Introduction

Definition 4.1. An isonormal Gaussian process $W = (W(h))_{h\in\mathfrak{H}}$ over $\mathfrak{H}$ defined on a complete probability space $(\Omega, \mathfrak{F}, \mathrm{P})$ is called *spatially homogeneous noise* on $\mathbb{R}_+ \times \mathbb{R}^d$ if there is a nonnegative definite tempered Borel measure Λ on $\mathbb{R}^d$ such that

$$\mathrm{E}\left[W(h)W(g)\right] = \langle h, g\rangle_{\mathfrak{H}}$$

for all $h,g \in \mathfrak{H}$ where $\mathfrak{H}$ is the the completion of the set of Schwartz functions $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to the inner product

$$\langle h,g\rangle_{\mathfrak{H}} := \int_0^\infty \int_{\mathbb{R}^d} \left(h(s,\cdot)\bar{*}g(s,\cdot)\right)(y)\Lambda(dy)ds$$

where

$$\left(h(s,\cdot)\bar{*}g(s,\cdot)\right)(y) = \int_{\mathbb{R}^d} h(s,x)g(s,x-y)dx.$$

We will call this measure Λ the *spectral measure* of the noise. If Λ is the Dirac mass at 0, then we call W *white noise* on $\mathbb{R}_+ \times \mathbb{R}^d$. If Λ is absolutely continuous with respect to Lebesgue measure, we will write $\Lambda(dx) = \lambda(x)dx$. Let, also, $\mathfrak{H}_0$ be the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the inner product

$$\langle h,g\rangle_{\mathfrak{H}_0} := \int_{\mathbb{R}^d} \left(h\bar{*}g\right)(y)\Lambda(dy)$$

and $\mathscr{B}_b(\mathbb{R}^d) := \{A \in \mathscr{B}(\mathbb{R}^d) : \|\mathbf{1}_A\|_{\mathfrak{H}_0} < \infty\}$.

Now, let W be a spatially homogeneous noise on $\mathbb{R}_+ \times \mathbb{R}^d$ with a spectral measure Λ and consider the random field $(W_t(A))_{(t,A)\in\mathbb{R}_+\times\mathscr{B}_b(\mathbb{R}^d)}$ defined as follows:

$$W_t(A) := W(1_{[0,t]\times A}), \text{ for } (t,A) \in \mathbb{R}_+ \times \mathscr{B}_b(\mathbb{R}^d). \tag{4.1}$$

Let further $(\mathfrak{F}_t)_{t\in\mathbb{R}_+}$ be the filtration such that

$$\mathfrak{F}_t = \mathfrak{F}_t^0 \vee \mathscr{N}, \;\; \mathfrak{F}_t^0 := \sigma(W_s(A), s \le t, A \in \mathscr{B}_b(\mathbb{R}^d)) \tag{4.2}$$

where $\mathscr{N}$ is the σ-field generated by P-null sets. By construction $(W_t(A))_{(t,A)\in\mathbb{R}_+\times\mathscr{B}_b(\mathbb{R}^d)}$ is a centred Gaussian random field.

Lemma 4.2. Let W be a spatially homogeneous noise on $\mathbb{R}_+ \times \mathbb{R}^d$ with spectral measure Λ. Then the centered Gaussian random field $(W_t(A))_{(t,A)\in\mathbb{R}_+\times\mathscr{B}_b}$ defined in (4.1) is a Gaussian martingale measure. That is to say for all $A,B\in\mathscr{B}_b$, we have

(i) $W_0(A)=0$;

(ii) $(W_t(A))_{t\in\mathbb{R}_+}$ is a continuous martingale with respect to the filtration $(\mathfrak{F}_t)_{t\in\mathbb{R}_+}$ defined in (4.2);

(iii) $\mathrm{E}\left[W_s(A)W_t(B)\right]=(s\wedge t)\int_{\mathbb{R}^d}(\mathbf{1}_A\tilde{*}\mathbf{1}_B)(y)\Lambda(dy)=(s\wedge t)\langle\mathbf{1}_A,\mathbf{1}_B\rangle_{\mathfrak{H}_0}$.

Proof. (i) Since $\mathrm{E}\left[(W_0(A))^2\right]=\mathrm{E}\left[\left(W(\mathbf{1}_{\{0\}\times A})\right)^2\right]=0$, it follows that $W_0(A)=0$ P-a.s. (ii) This follows from the definition of the filtration. (iii) This also follows quickly from the definition of W since

$$\mathrm{E}\left[W_s(A)W_t(B)\right]=\int_{\mathbb{R}_+}\int_{\mathbb{R}^{2d}}\mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,s]}(r)\mathbf{1}_A(x)\mathbf{1}_B(x-y)dx\Lambda(dy)dr.$$

□

Lemma 4.3. Let W be a spatially homogeneous noise on $\mathbb{R}_+\times\mathbb{R}^d$ with spectral measure Λ and $(W_t(A))_{(t,A)\in\mathbb{R}_+\times\mathscr{B}_b(\mathbb{R}^d)}$ be the corresponding Gaussian martingale measure. Let $(A_n)_{n\in\mathbb{N}}\subset\mathscr{B}_b(\mathbb{R}^d)$ be pairwise disjoint sets with $\cup_{n\in\mathbb{N}}A_n\in\mathscr{B}_b(\mathbb{R}^d)$. Then for all $t\in\mathbb{R}_+$:

$$W_t(\cup_{n\in\mathbb{N}}A_n)=\sum_{n\in\mathbb{N}}W_t(A_n),$$

where the series on the left converges in $L^2(\Omega,\mathfrak{F},\mathrm{P})$.

Proof. Let $A:=\cup_{n\in\mathbb{N}}A_n$ and $B_N:=\cup_{n=1}^N A_n$. For $N\in\mathbb{N}$, we have

$$\begin{aligned}\mathrm{E}\left[\left(\sum_{n=1}^N W_t(A_n)-W_t(A)\right)^2\right]&=\mathrm{E}\left[\sum_{n,m=1}^N W_t(A_n)W_t(A_m)-2W_t(A)\sum_{n=1}^N W_t(A_n)+W_t(A)^2\right]\\&=\sum_{n,m=1}^N t\langle\mathbf{1}_{A_n},\mathbf{1}_{A_m}\rangle_{\mathfrak{H}_0}-2t\sum_{n=1}^N\langle\mathbf{1}_A,\mathbf{1}_{A_n}\rangle_{\mathfrak{H}_0}+t\langle\mathbf{1}_A,\mathbf{1}_A\rangle_{\mathfrak{H}_0}\\&=t\langle\mathbf{1}_{B_N},\mathbf{1}_{B_N}\rangle_{\mathfrak{H}_0}-2t\langle\mathbf{1}_A,\mathbf{1}_{B_N}\rangle_{\mathfrak{H}_0}+t\langle\mathbf{1}_A,\mathbf{1}_A\rangle_{\mathfrak{H}_0}.\end{aligned}$$

Since $\mathbf{1}_{B_N} \nearrow \mathbf{1}_A$, by monotone convergence theorem, we have

$$\lim_{N\to\infty} \mathrm{E}\left[\left(\sum_{n=1}^{N} W_t(A_n) - W_t(A)\right)^2\right] = 0.$$

$\square$

Examples 4.4. (i) An important example of a spatially homogeneous noise on $\mathbb{R}_+ \times \mathbb{R}^d$ is *the space-time white noise* where $\Lambda(dx) = \delta_0(dx)$ with the corresponding Hilbert space $\mathfrak{H} = L^2(\mathbb{R}_+ \times \mathbb{R}^d, \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}^d), m)$.

(ii) Another important example is where $\Lambda(dx) = \lambda(x)dx$, $\lambda(x) = |x|^{-\beta}$ for $\beta \in [0,d)$ called Riesz kernel.

(iii) For a given spatially homogeneous noise, we can construct other spatially homogeneous noises as follows: For $\varphi \in C_c^\infty(\mathbb{R}^d)$, let

$$W^\varphi(h) := W(h *_x \varphi),$$

where $*_x$ corresponds to convolution in space variable.

Definition 4.5. A random field $X = \{X(s,y)\}$ on a complete probability space $(\Omega, \mathfrak{F}, \mathrm{P})$ is *elementary* if

$$X(s,y) = F\mathbf{1}_{(a,b]}(s)\mathbf{1}_A(y), \tag{4.3}$$

where $0 \leq a < b$, $A \in \mathscr{B}_b(\mathbb{R}^d)$ and F is a bounded, $\mathfrak{F}_a$-measurable random variable. Let $\mathscr{E}(\mathbb{R}_+ \times \mathbb{R}^d \times \Omega)$ denote the set of all finite linear combinations of elementary random fields.

Let $\mathscr{P}$ be the smallest σ-algebra on $\mathbb{R}_+ \times \mathbb{R}^d \times \Omega$ such that all $X \in \mathscr{E}$ is measurable. A function $X : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is called *predictable* if it is measurable with respect to $\mathscr{P}$. For a predictable process X, define the norm

$$\|X\|_{L^2(W)}^2 := \mathrm{E}\left[\int_0^\infty \int_{\mathbb{R}^d} |(X(s,\cdot)\tilde{*}X(s,\cdot))(y)|\,\Lambda(dy)ds\right]$$

and the space

$$L^2(W) := \left\{X : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R};\, X \text{ is predictable and } \|X\|_W < \infty\right\},$$

where we identify $X, X' \in L^2(W)$ such that $\|X - X'\|_W = 0$.

Proposition 4.6. (i) $L^2(W)$ is a Banach space .

(ii) $\mathscr{E} \cap L^2(W)$ is dense in $L^2(W)$.

Proof. See Walsh [56, Proposition 2.6] and Conus [18, Theorem 2.6]. □

Definition 4.7. For an elementary random field X of the form (4.3), define

$$\int_{[0,t]\times\mathbb{R}^d} X(s,y)W(ds,dy) = F(W_{b\wedge t}(A) - W_{a\wedge t}(A)). \tag{4.4}$$

Proposition 4.8. Let $X \in \mathscr{E} \cap L^2(W)$. Then for $t \geq 0$

$$M_t := \int_{[0,t]\times\mathbb{R}^d} X(s,y)W(ds,dy)$$

is a continuous martingale with quadratic variation

$$\langle M\rangle_t = \int_0^t \int_{\mathbb{R}^d} (X(s,\cdot)\tilde{*}X(s,\cdot))(y)\Lambda(dy)ds.$$

In particular, Itô isometry holds:

$$\mathrm{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}^d} X(s,y)W(ds,dy)\right)^2\right] = \mathrm{E}\left[\int_0^\infty \int_{\mathbb{R}^d} (X(s,\cdot)\tilde{*}X(s,\cdot))(y)\Lambda(dy)ds\right] \leq \|X\|_W^2.$$

Proof. We will only consider the case where X is elementary. The case for $X \in \mathscr{E} \cap L^2(W)$ follows by linearity. Let $X(t,x)$ be given as in (4.3). Then $M_t = F(W_{b\wedge t}(A) - W_{a\wedge t}(A))$ being continuous martingale and the formula

$$
\begin{aligned}
\langle M \rangle_t &= \int_0^t \int_{\mathbb{R}^d} F^2 \mathbf{1}_{[a,b)}(s)(\mathbf{1}_A \tilde{*} \mathbf{1}_A)(y)\Lambda(dy)ds \\
&= \int_0^t \int_{\mathbb{R}^d} (X(s,\cdot)\tilde{*}X(s,\cdot))(y)\Lambda(dy)ds
\end{aligned}
$$

follows from Lemma 4.2 together the fact that F is $\mathfrak{F}_a$-measurable.

□

Let $\mathscr{M}_c^2$ be the family of uniformly integrable continuous martingales M such that $M_0 = 0$ and $\mathrm{E}\left[M_\infty^2\right] < \infty$. After identifying two martingales if they are indistinguishable, $\mathscr{M}_c^2$ is a Hilbert space with the inner product

$$
(M,N)_{\mathscr{M}_c^2} := \mathrm{E}\left[M_\infty N_\infty\right] = \mathrm{E}\left[\langle M,N \rangle_\infty\right].
$$

Define the map

$$
\begin{aligned}
I_W : \mathscr{E} \cap L^2(W) &\to \mathscr{M}_c^2, \\
X &\mapsto \int_{[0,\cdot]\times\mathbb{R}^d} X(s,y)W(ds,dy)
\end{aligned}
\tag{4.5}
$$

which is linear by construction. Moreover, from previous proposition we have that for $X \in \mathscr{E} \cap L^2(W)$,

$$
\|I_W(X)\|_{\mathscr{M}_c^2} \le \|X\|_{L^2(W)},
$$

which implies I_W is Lipschitz.

Theorem 4.9. The map J_W uniquely extends to a linear map from $L^2(W)$ to $\mathscr{M}_c^2$ which we still denote by J_W. Moreover, Itô's isometry holds: For all $t \in [0,\infty]$ and $X \in L^2(W)$, we have

$$\mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}^d} X(s,y)W(ds,dy)\right)^2\right] = \mathrm{E}\left[\int_0^t \int_{\mathbb{R}^d} (X(s,\cdot)\tilde{*}X(s,\cdot))(y)\Lambda(dy)ds\right] \leq \|\mathbf{1}_{[0,t]}X\|_W^2. \quad (4.6)$$

We will call $J_W(X) = \int_{[0,\cdot]\times\mathbb{R}^d} X(s,y)W(ds,dy)$ the stochastic integral of X with respect to W.

Proof. By Proposition 4.6 (ii), we know $\mathscr{E}\cap L^2(W)$ is dense in $L^2(W)$. For given $X \in L^2(W)$, let $(X_n)_{n\in\mathbb{N}} \subset \mathscr{E}\cap L^2(W)$ be a sequence converging to X in $L^2(W)$. Then $(J_W(X_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{M}_c^2$ by Itô-Walsh isometry in Proposition 4.8 and therefore has a limit, denoted $J_W(X) = \int_{[0,\cdot]\times\mathbb{R}^d} X(s,y)W(ds,dy)$. This definition is independent of the approximating sequence. Indeed, using Itô isometry and Lipschitz property of J_W, this can easily be verified. Linearity, Itô's isometry, and Lipschitz continuity follows similarly. □

The following result follows applying the usual Burkholder-Davis-Gundy inequality for the continuous martingales to the martingale $M_t = \int_{[0,t]\times\mathbb{R}^d} X(s,y)W(ds,dy)$ with quadratic variation $\langle M\rangle_t = \int_0^t \int_{\mathbb{R}^d} (X(s,\cdot)\tilde{*}X(s,\cdot))(y)\Lambda(dy)ds$.

Theorem 4.10. For all $p > 0$ there extists a constant $C_p > 0$ such that for all $X \in L^2(W)$ and for all $t \in [0,\infty]$:

$$\begin{aligned}\frac{1}{C_p}\mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}^{2d}} X(s,x)X(s,x-y)dx\Lambda(dy)ds\right)^{p/2}\right] &\leq \mathrm{E}\left[\sup_{s\in[0,t]}\left|\int_{[0,s]\times\mathbb{R}^d} X(r,x)W(dr,dx)\right|^p\right] \\ &\leq C_p\mathrm{E}\left[\left(\int_0^t \int_{\mathbb{R}^{2d}} X(s,x)X(s,x-y)dx\Lambda(dy)ds\right)^{p/2}\right]\end{aligned}$$

4.2 Walsh integral and Malliavin calculus

The results in this section are extensions of Theorem 2.47 and Theorem 2.48, for spatially homogeneous noise first obtained in [16].

Let W be the spatially homogeneous noise with spectral measure Λ and D, δ be the Malliavin derivative and divergence operators introduced in first chapter corresponding to W. From this construction, we see that $\mathfrak{H} = L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), m; \mathfrak{H}_0)$. We will use the identification

$$L^2(\Omega, \mathfrak{F}, \mathrm{P}; L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), m; \mathfrak{H}_0)) \cong L^2(\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathfrak{F}, m \otimes \mathrm{P}; \mathfrak{H}_0).$$

Thus the Malliavin derivative can be viewed as a stochastic process $\{D_t X : t \in \mathbb{R}_+\}$ taking values in $\mathfrak{H}_0$. For an elementary process V of the form (4.3), recall that the Walsh integral is given by

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} V(s,y) W(ds, dy) = F(W_b(A) - W_a(A)) = FW(\mathbf{1}_{[a,b) \times A}).$$

V can be seen as a process taking values in $\mathfrak{H}_0$ as follows:

$$V(t) = F \mathbf{1}^A_{[a,b)}(t)$$

where

$$\mathbf{1}^A_{[a,b)} : \mathbb{R}_+ \to \mathfrak{H}_0, \quad \mathbf{1}^A_{[a,b)}(t) \mapsto \mathbf{1}_{[a,b)}(t) \mathbf{1}_A.$$

Theorem 4.11. Let $V \in L^2(W)$, then $V \in \mathrm{Dom}\,(\delta)$ as an $\mathfrak{H}_0$-valued process, moreover

$$\delta(V) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} V(s,y) W(ds, dy)$$

for any $V \in L^2(W)$.

Proof. Let V be an elementary random field of the form (2.8) where $F \in \mathscr{S}$. Using integration by parts (2.5), we see

$$\mathrm{E}\left[\langle V, DG \rangle_{\mathfrak{H}}\right] = \mathrm{E}\left[F \langle \mathbf{1}^A_{[a,b)}, DG \rangle_{\mathfrak{H}}\right] = \mathrm{E}\left[FGW(\mathbf{1}^A_{[a,b)}) - G \langle \mathbf{1}^A_{[a,b)}, DF \rangle_{\mathfrak{H}}\right]. \tag{4.7}$$

Note that since $F \in \mathscr{S}$ is $\mathfrak{F}_a$-measurable, $F = f(W(h_1), \cdots, W(h_n))$ for some smooth f and $h_i \in L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), m; \mathfrak{H}_0)$ such that supp $h_i \subset [0,a]$. This implies, in particular, $\langle \mathbf{1}^A_{[a,b)}, h_i \rangle = 0$ for all

$i=1,\cdots m$ and $\langle \mathbf{1}^A_{[a,b)}, DF\rangle_{\mathfrak{H}}=0$. So, the above identity becomes

$$\mathrm{E}\left[\langle V, DG\rangle_{\mathfrak{H}}\right]=\mathrm{E}\left[FGW(\mathbf{1}^A_{[s,t)}\right]$$

which can be rewritten using (4.7) as

$$\mathrm{E}\left[\langle V, DG\rangle_{\mathfrak{H}}\right]=\mathrm{E}\left[G\int_{\mathbb{R}_+\times\mathbb{R}^d} V(s,y)W(ds,dy)\right].$$

Then the proof can be completed by an approximation argument. $\square$

The following is the extension of Clark-Ocone formula 2.48 for spatially homogeneous noise.

Theorem 4.12. For every $F\in\mathbb{D}^{1,2}$,

$$F=\mathrm{E}[F]+\int_{\mathbb{R}_+\times\mathbb{R}^d}\mathrm{E}[D_{s,y}F|\mathfrak{F}_s]\,W(ds,dy),\ \text{ a.s.}$$

Consequently, Poincare inequality holds:

$$\mathrm{Var}[F]\le \mathrm{E}\left[\|DF\|^2_{\mathfrak{H}}\right]$$

Proof. Let $F\in\mathbb{D}^{1,2}$ be given. Since one can extend the martingale representation theorem to martingales taking values in a Hilbert space, it follows that there is an adapted random field $U(s,y)$ such that

$$F=\mathrm{E}[F]+\int_{\mathbb{R}_+\times\mathbb{R}^d} U(s,y)W(ds,dy). \tag{4.8}$$

We want to show that $U(s,y)=\mathrm{E}\left[D_{s,y}F\middle|\mathfrak{F}_s\right]$ as elements in $L^2_a(\mathbb{R}_+\times\Omega,\mathscr{B}(\mathbb{R}^d)\otimes\mathfrak{F},m\otimes\mathrm{P};\mathfrak{H}_0)$. Let $V\in L^2_a(\mathbb{R}_+\times\Omega,\mathscr{B}(\mathbb{R}^d)\otimes\mathfrak{F},m\otimes\mathrm{P};\mathfrak{H}_0)$. On one hand, using the isometry property of Walsh integral (4.6), we see

$$\mathrm{E}\left[\delta(V)F\right]=\mathrm{E}\left[\langle U,V\rangle_{\mathfrak{H}}\right].$$

On the other hand, using integration by parts (2.7), we get

$$\begin{aligned}\mathrm{E}\left[\delta(V)F\right]=\mathrm{E}\left[\langle V,DF\rangle_{\mathfrak{H}}\right]&=\mathrm{E}\left[\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}(V(s,\cdot)\bar{*}D_{s,\cdot}F)(y)\Lambda(dy)ds\right]\\&=\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\mathrm{E}\left[\left(V(s,\cdot)\bar{*}\mathrm{E}\left[D_{s,\cdot}F\middle|\mathfrak{F}_s\right]\right)\right](y)\Lambda(dy)ds\end{aligned}$$

where we used the fact that V is adapted to the filtration $\{\mathfrak{F}_t\}_{t\in\mathbb{R}_+}$. The above findings together verifies the claim.

□

Remark 4.13. In dimension 1 and the case where $\Lambda=\delta_0$, the operators D and δ satisfy the following commutation relation

$$D_{s,y}(\delta(V))=V(s,y)+\delta(D_{s,y}V), \tag{4.9}$$

for almost all $(s,y)\in\mathbb{R}_+\times\mathbb{R}$, provided $V\in\mathbb{D}^{1,2}(\mathfrak{H})$ is such that for almost all $(s,y)\in\mathbb{R}_+\times\mathbb{R}$, $D_{s,y}V$ belongs to the domain of the divergence in L^2 and $\mathrm{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}}|\delta(D_{s,y}V)|^2dsdy\right]<\infty$ (see [42, Proposition 1.3.2]).

Chapter 5

Stochastic heat equation

Stochastic partial differential equations (SPDEs) are mathematical objects that model physical phenomena under influence of random noise and stochastic heat equation (SHE) is one important example. Among different approaches to solving such equations we concentrate on the random field approach, in which the multi-parameter stochastic integral that we have introduced in the previous chapter can be viewed as a continuation of Itô's stochastic calculus is used. This approach is pioneered in Walsh's lecture notes [56]. In particular, we focus on the stochastic heat equation governed by a spatially homogeneous noise. In the first half of this chapter, we establish the existence and regularity of the solution using the references Chen [10], Dalang [20], Dalang, Khoshnevisan, Mueller, Nualart, and Xiao [21], Hairer [22], Khoshnevisan [28], Perkowski [50], Walsh [56]. We then establish Malliavin differentiability of the solution in the second part for which we refer to the papers Chen and Huang [12], Chen and Kim [13], Chen, Hu, and Nualart [14], Kuzgun and Nualart [32].

5.1 Existence and regularity

Let W be a spatially homogeneous noise in $\mathbb{R}_+ \times \mathbb{R}^d$ with spectral measure Λ. Consider

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + g(u)\dot{W}, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \\ u(0,x) = u_0, \end{cases} \tag{5.1}$$

where $d \in \mathbb{N}$, and $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator. The initial condition u_0 is in general assumed to be a signed Borel measure on $\mathbb{R}^d$ such that for all $c > 0$,

$$\int_{\mathbb{R}^d} e^{-c|x|^2} |u_0|(dx) < \infty \tag{5.2}$$

and g is a nonrandom Lipschitz function with Lipschitz constant Lip_g.

Definition 5.1. We say that a nonnegative, nonnegative definite, tempered, Borel measure Λ on $\mathbb{R}^d$ satisfy *Dalang's condition* with $r \in (0,1]$ if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^r} < \infty \tag{5.3}$$

where Λ is the Fourier transform of a tempered Borel measure μ. In other words: for $h, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle h, g \rangle_{\mathfrak{H}_0} := \int_{\mathbb{R}^d} (h \tilde{*} g)(y) \Lambda(dy) = \int_{\mathbb{R}^d} h(\xi) \overline{g(\xi)} \mu(d\xi).$$

Examples 5.2. (i) Recall space-time white noise in Examples 4.4. $\Lambda(dx) = \delta_0(dx)$ satisfies Dalang's condition for all $r \in (0,1]$ if $d = 1$. Otherwise, it doesn't satisfy the Dalang's condition. This can be seen by noting that $\mu(d\xi) = d\xi$.

(ii) Recall the noise W with the spectral measure $\lambda(x) = |x|^{-\beta}$ Examples 4.4. Then, W satisfies the Dalang's condition with any $r < \beta/2 \in (0,1)$ if $\beta < \min(2,d)$. Note that in this case $\mu(d\xi) = |\xi|^{d-\beta} d\xi$.

Remark 5.3. Λ is the Fourier transform of a tempered measure μ on $\mathbb{R}^d$ follows from Bochner-Schwartz theorem. See [53].

Now, we define the notion of the mild and weak solutions to the SHE (5.1) using the stochastic integral we introduced in previous section. Under some conditions it turns out that these two definitions are equivalent, see for example [48, Proposition 3.2], [22, Proposition 5.7]. For the sake of completeness, we will give both definitions but will only use the mild solution in the rest of this thesis.

Definition 5.4. A predictable random field $u = \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ is called *weak solution* to the stochastic heat equation in (5.1) if for all $\varphi \in C_c(\mathbb{R}^d)$ the following integrals are well-defined

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} \varphi(x)u_0(dx) + \int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)dxds + \int_{[0,t]\times\mathbb{R}^d} g(u(s,x))\varphi(x)W(ds,dx). \tag{5.4}$$

Notation 5.5. For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, set

$$\boldsymbol{p}_t(x) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} \tag{5.5}$$

which we call *heat kernel*.

Definition 5.6. A predictable random field $u = \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ is called *mild solution* to the stochastic heat equation in (5.1) if all of the following integrals are well-defined and for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$u(t,x) = \int_{\mathbb{R}^d} \boldsymbol{p}_t(x-y)u_0(dy) + \int_{[0,t]\times\mathbb{R}^d} \boldsymbol{p}_{t-s}(x-y)g(u(s,x))W(ds,dx). \tag{5.6}$$

Let τ_z denote the translation operator for $z \in \mathbb{R}^d$ and let $h_z \in \mathfrak{H}$ be such that $h_z := \tau_z(h)$.

Lemma 5.7. Let W be a spatially homogeneous noise on $\mathbb{R}_+ \times \mathbb{R}^d$ with spectral measure Λ. Let $z \in \mathbb{R}^d$. Then for any $h \in \mathfrak{H}$, $W(h)$ and $W_z(h) := W(h_z)$ has the same law.

Proof. It enough to show that for all $z \in \mathbb{R}^d$, the covariances agree, that is $\langle h,g\rangle_{\mathfrak{H}} = \langle h_z, g_z\rangle_{\mathfrak{H}}$. Indeed if $h,g \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)$, this claim follows from the fact that $(h(t,\cdot)\bar{*}g(t,\cdot))(y) = (h_z(t,\cdot)\bar{*}g_z(t,\cdot))(y)$ for all $y \in \mathbb{R}^d$. $\square$

Lemma 5.8. Let $X(s,y)$ be a random field such that for all $y,z \in \mathbb{R}^d$

$$g_s(z) := \mathrm{E}[X(s,y)X(s,y+z)]$$

is independent of y and $p_{t-\cdot}(x-\cdot)u(\cdot,\cdot) \in L^2(W)$, then so

$$\mathrm{E}\left[\left(\int_{[0,t]\times\mathbb{R}^d} p_{t-s}(x-y)X(s,y)W(ds,dy)\right)\left(\int_{[0,t]\times\mathbb{R}^d} p_{t-s}(x+z-y)X(s,y)W(ds,dy)\right)\right]$$

is independent of x.

Proof. Following the ideas in [16] and Lemma 5.7, we see that

$$\int_{[0,t]\times\mathbb{R}^d} p_{t-s}(x+z-y)X(s,y)W(ds,dy) = \int_{[0,t]\times\mathbb{R}^d} p_{t-s}(x-y)X(s,y+z)W_z(ds,dy)$$

has the same distribution as

$$\int_{[0,t]\times\mathbb{R}^d} p_{t-s}(x-y)X(s,y+z)W(ds,dy).$$

Then result follows by the assumption, together with Itô-Walsh isometry. $\square$

Lemma 5.9. Let Λ be a measure satisfying Dalang's condition (5.3) with some $r \in (0,1]$. Then, for all $T > 0$ there exists $0 < C_{T,r} < \infty$ such that

$$\int_0^T \int_{\mathbb{R}^d} (\boldsymbol{p}_t * \boldsymbol{p}_t)(z)\Lambda(dz)dt \le C_{T,r} \tag{5.7}$$

Proof. Note that the spatial Fourier transform of the heat kernel is

$$\hat{\boldsymbol{p}}_t(\xi) = e^{-t|\xi|^2}.$$

Then, using the properties of Fourier transform, we have

$$\int_{\mathbb{R}^d} (\boldsymbol{p}_t * \boldsymbol{p}_t)(z)\Lambda(dz) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t|\xi|^2}\mu(d\xi)$$

and

$$\int_0^T \int_{\mathbb{R}^d} (\boldsymbol{p}_t * \boldsymbol{p}_t)(z)\Lambda(dz)dt = \int_{\mathbb{R}^d}\int_0^T e^{-i\xi\cdot x} e^{-t|\xi|^2}\mu(d\xi)dt = \int_{\mathbb{R}^d} \frac{1-e^{-T|\xi|^2}}{|\xi|^2}\mu(d\xi)$$

Now, multiplying and dividing by $(1+|\xi|^2)^r$ using Dalang's condition (5.3) on μ, we see

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} (\boldsymbol{p}_t * \boldsymbol{p}_t)(x-z)\Lambda(dz) &\le C_{T,r}\left(\int_{|\xi|\le 1}\mu(d\xi) + \int_{|\xi|>1}\frac{\mu(d\xi)}{(1+|\xi|^2)^r}\right)\\ &\le C_{T,r} < \infty\end{aligned}$$

which completes the proof. □

Theorem 5.10. Let W be a spatially homogeneous noise on $\mathbb{R}_+\times\mathbb{R}^d$ with a spectral measure Λ which satisfies Dalang's condition with $r\in(0,1]$. Let u_0 be a bounded function on $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d))$. Assume g is a Lipschitz function. Then there exists a unique predictable process u satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \le C_{T,p}.$$

which is a mild solution to (5.1).

Proof. We will only consider the case $u_0\equiv 0$. We will follow the Picard iteration: Let $u_0(t,x)=0$ and assuming that u_n has been defined as a L^2-bounded random field such that $u_n(t,x)$ is $\mathfrak{F}_t$ measurable and $L^2(\Omega,\mathfrak{F},\mathrm{P})$-continuous, set

$$u_{n+1}(t,x) = \int_{[0,t]\times\mathbb{R}^d}\boldsymbol{p}_{t-s}(x-y)g(u_n(s,y))W(ds,dy). \tag{5.8}$$

Then, u_{n+1} as defined in (5.8) is well-defined, L^2-bounded, $L^2(\Omega,\mathfrak{F},\mathrm{P})$ continuous, $\mathfrak{F}_t$-measurable.

See [20, Theorem 13] or [10, Proposition 3.3.4].

Now, we claim that $(u_n(t,x))_n$ converges in $L^P(\Omega,\mathfrak{F},\mathrm{P})$, uniformly in $(t,x)\in[0,T]\times\mathbb{R}^d$. To prove this, let

$$f_n(t)=\sup_{x\in\mathbb{R}^d}\mathrm{E}\left[|u_{n+1}(t,x)-u_n(t,x)|^p\right].$$

By the isometry property of the Walsh integral (4.6) and using g being Lipschitz together with Lemma 5.8, we obtain

$$f_n(t)\le C_g\int_0^t f_n(s)G(t-s)ds,$$

where

$$\begin{aligned}G(s):&=\int_{\mathbb{R}^{2d}}\boldsymbol{p}_s(x-y+y')\boldsymbol{p}_s(x-y)dy\Lambda(dy')\\&=\int_{\mathbb{R}^d}\boldsymbol{p}_{2s}(y')\Lambda(dy'),\end{aligned}$$

where we used semigroup property. Since G is a nonnegative, integrable function on $[0,T]$, we can apply Gronwall's Lemma A.2, we obtain

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\mathrm{E}\left[|u_m(t,x)-u_n(t,x)|^p\right]\le\sum_{k=n+1}^m a_k^{1/p}\to 0$$

as $m,n\to\infty$. Uniqueness also follows similarly. $\square$

The following version covers the delta initial condition, see [11] for a proof.

Theorem 5.11. Let W be a spatially homogeneous noise on $\mathbb{R}_+\times\mathbb{R}^d$ with a spectral measure Λ which satisfies Dalang's condition with $r\in(0,1]$. Let $u_0=\delta_0$ be the Dirac mass at 0. Assume g is a Lipschitz function. Then there exists a unique predictable process u which satisfies

$$\|u(t,x)\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})}\le C_{T,p}\boldsymbol{p}_t(x)$$

and a mild solution to (5.1).

See [10] and [11] for the following result.

Theorem 5.12. Let W be a spatially homogeneous noise on $\mathbb{R}_+ \times \mathbb{R}^d$ with a spectral measure Λ which satisfies Dalang's condition with $r \in (0,1)$. Let u be the mild solution to (5.1). Then for all $p \geq 2$, $\gamma_1 \in (0, \frac{1-r}{2}), \gamma_2 \in (0, 1-r)$, there exists a constant C such that for all $s,t \in [0,T]$

$$\|u(t,x) - u(s,y)\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq C\left(|t-s|^{\gamma_1} + |x-y|^{\gamma_2}\right) + |I_0(t,x) - I_0(s,y)|$$

where we used the notation

$$I_0(t,x) := \int_{\mathbb{R}^d} p_t(x-\xi)u_0(d\xi).$$

Corollary 5.13. Let W be a white noise on $\mathbb{R}_+ \times \mathbb{R}$ and u be the mild solution to (5.1) with initial condition $u_0 \equiv 1$ or $u_0 \equiv \delta_0$. Then for all $p \geq 2$, there exists a constant C such that for all $s,t \in [0,T]$

$$\|u(t,x) - u(s,y)\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq C\left(|t-s|^{1/4} + |x-y|^{1/2}\right).$$

See [12] and [14] for the following result.

Theorem 5.14. Let W be a spatially homogeneous noise on $\mathbb{R}_+ \times \mathbb{R}^d$ with a spectral measure Λ which satisfies Dalang's condition with $r \in (0,1]$, and u be the mild solution to (5.1) with nonnegative initial condition $u_0 > 0$ satisfying (5.2). Further assume g is a Lipschitz function such that $g(0) = 0$. Then for all $p > 0$, $K \subset \mathbb{R}^d$ compact, and $t > 0$:

$$\mathrm{E}\left[\left(\inf_{x\in K} u(t,x)\right)^{-p}\right] < \infty.$$

Furthermore,

(a) If W is space-time white noise in dimension $d=1$, $g(x)=x$ and $u_0(x)=1$, then

$$\mathrm{E}\left[\left(\inf_{(t,x)\in K} u(t,x)\right)^{-p}\right] < \infty$$

for any $K \subset \mathbb{R}_+ \times \mathbb{R}$ compact.

(b) If W is space-time white noise in dimension $d=1$, $g(x)=x$ and $u_0(x)=\delta_0(x)$, then

$$\mathrm{E}\left[\left(\inf_{t\in K} u(t,0)\right)^{-p}\right] < \infty$$

for any $K \subset \mathbb{R}_+$ compact.

5.2 Malliavin differentiblity

5.2.1 Parabolic Anderson model

In this subsection, we will consider the case where $g(u)=u$. Namely, the equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}, \ x\in\mathbb{R}^d, t\in\mathbb{R}_{>0}, \\ \\ u(0,x)=u_0, \end{cases} \tag{5.9}$$

with initial condition u_0 is assumed to be a signed Borel measure on $\mathbb{R}^d$ satisfying (5.2).

Proposition 5.15. For any $(t,x)\in(0,\infty)\times\mathbb{R}^d$, $u(t,x)\in\mathbb{D}^\infty$.

Proof. From Part (2) of [12, Proposion 3.2] it follows that $u(t,x)\in\mathbb{D}^{1,p}$ for all $(t,x)\in(0,\infty)\times\mathbb{R}^d$ and for all $p\geq 1$. Because we are dealing with the parabolic Anderson model, the proof of Part (3) of [12, Proposion 3.2] implies that $u(t,x)\in\mathbb{D}^\infty$ for all $(t,x)\in(0,\infty)\times\mathbb{R}^d$. $\square$

When we are in this specific case, there are more tools to work with to obtain some properties of the solution. One of these is Feynman-Kac formulas. The purpose of this section is to obtain such formulas for the moments of the solution and its derivatives. These formulas will then be used to get estimates for the p-th norm of the first and higher order derivatives which are important in the application of Malliavin-Stein method.

Keeping this motivation in mind, we will now introduce an approximation scheme for the homogeneous noise that we will then use to approximate the solution to the parabolic Anderson model (5.9).

For each $\varepsilon > 0$ and any $\varphi \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)$, we define (recall from Examples 4.4)

$$W^{\varepsilon}(\varphi) = W(\varphi(t,\cdot) * \boldsymbol{p}_{\varepsilon}(\cdot)),$$

where $*$ denotes the convolution in the space variable and $\boldsymbol{p}_{\varepsilon}(x)$ is the d-dimensional heat kernel defined in (5.5). Then, the Gaussian family $W^{\varepsilon} = \left\{W^{\varepsilon}(\varphi); \varphi \in \mathcal{S}\left(\mathbb{R}_+ \times \mathbb{R}^d\right)\right\}$ has the covariance structure

$$\begin{aligned}
&\mathrm{E}\left[W^{\varepsilon}(\varphi)W^{\varepsilon}(\psi)\right] \\
&= \int_0^\infty \int_{\mathbb{R}^{2d}} \left(\varphi(s,\cdot) * \boldsymbol{p}_{\varepsilon}(\cdot)\right)(y) \left(\psi(s,\cdot) * \boldsymbol{p}_{\varepsilon}(\cdot)\right)(y-y')\Lambda(dy')dyds \\
&= \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}(\varphi)(s,\xi)\overline{\mathcal{F}(\psi)(s,\xi)}e^{-\varepsilon|\xi|^2}\mu(d\xi)ds,
\end{aligned}$$

that is, the noise W^{ε} is white in time and it has a spatial covariance given by

$$\lambda_{\varepsilon}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi-\varepsilon|\xi|^2}\mu(d\xi), \tag{5.10}$$

whose Fourier transform is $\mu_{\varepsilon}(d\xi) = e^{-\varepsilon|\xi|^2}\mu(d\xi)$. Notice that μ_{ε} is a finite measure and λ_{ε} is a bounded smooth function. In this way, we can write

$$\begin{aligned}
\mathrm{E}\left[W^{\varepsilon}(\varphi)W^{\varepsilon}(\psi)\right] &= \int_0^\infty \int_{\mathbb{R}^{2d}} \varphi(s,y)\psi(s,y')\lambda_{\varepsilon}(y-y')dydy'ds \\
&= \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}(\varphi)(s,\xi)\overline{\mathcal{F}(\psi)(s,\xi)}\mu_{\varepsilon}(d\xi)ds.
\end{aligned}$$

As before, we denote by $\mathfrak{H}^{\varepsilon}$ the completion of $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)$ under the inner product

$$\langle \varphi, \psi \rangle_{\mathfrak{H}^{\varepsilon}} = \mathrm{E}\left[W^{\varepsilon}(\varphi) W^{\varepsilon}(\psi)\right].$$

Let $\phi^t : [0,t] \to \mathbb{R}^d$ be a continuous function for each $t \in \mathbb{R}_+$. Then, the map $(s,y) \mapsto \mathbf{1}_{[0,t]}(s) \boldsymbol{p}_{\varepsilon}\left(\phi^t(s) - y\right)$ belongs to the space $\mathfrak{H}$ since

$$\begin{aligned}
&\|\mathbf{1}_{[0,t]}(\bullet) \boldsymbol{p}_{\varepsilon}\left(\phi^t(\bullet) - \star\right)\|_{\mathfrak{H}}^2 \\
&\quad = \int_0^t \int_{\mathbb{R}^{2d}} \boldsymbol{p}_{\varepsilon}\left(\phi^t(s) - y\right) \boldsymbol{p}_{\varepsilon}\left(\phi^t(s) - y' + y\right) \Lambda(dy') dy ds \\
&\quad = \int_0^t \int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2} \mu(d\xi) ds = t \int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2} \mu(d\xi) \\
&\quad = t(2\pi)^d \lambda_{\varepsilon}(0) < \infty
\end{aligned} \tag{5.11}$$

and we can define the stochastic integral

$$W\left(\mathbf{1}_{[0,t]}(\bullet) \boldsymbol{p}_{\varepsilon}\left(\phi^t(\bullet) - \star\right)\right) = \int_{[0,t] \times \mathbb{R}^d} \boldsymbol{p}_{\varepsilon}(\phi^t(s) - y) W(ds, dy).$$

Throughout, we will use the following notation:

$$\int_0^t W^{\varepsilon}(ds, \phi^t(s)) := \int_{[0,t] \times \mathbb{R}^d} \boldsymbol{p}_{\varepsilon}(\phi^t(s) - y) W(ds, dy).$$

From (5.11) it follows that $\int_0^t W^{\varepsilon}(ds, \phi^t(s))$ is a centered Gaussian random variable with variance $t(2\pi)^d \lambda_{\varepsilon}(0)$.

Now, we consider the heat equation driven by W^{ε},

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + u \dot{W}^{\varepsilon}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{5.12}$$

with the same initial condition $u(0,x) = u_0$. An adapted and jointly measurable random field $u^{\varepsilon} = \{u^{\varepsilon}(t,x); (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ such that $\mathrm{E}\left[u^{\varepsilon}(t,x)\right]^2 < \infty$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ is a mild solution to equation (5.12), if for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process $\{\boldsymbol{p}_{t-s}(x-y)u^\varepsilon(s,y)\mathbf{1}_{[0,t]}(s);(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is integrable with respect to W^ε, and the following holds:

$$u^\varepsilon(t,x) = (\boldsymbol{p}_t * u_0)(x) + \int_{[0,t]\times\mathbb{R}^d} \boldsymbol{p}_{t-s}(x-y)u^\varepsilon(s,y)W^\varepsilon(ds,dy). \tag{5.13}$$

It follows from the general theory that this mild solution exists and it is unique. Furthermore, because the spectral measure is finite, there is a Feynman-Kac representation of the solution, given in the following lemma.

Lemma 5.16. For each $\varepsilon > 0$, the following random field $u^\varepsilon(t,x)$ is the solution to the heat equation given in (5.12):

$$u^\varepsilon(t,x) = \mathrm{E}^B\left[u_0(B_t^x)\exp\left(\int_0^t W^\varepsilon\left(ds,B_{t-s}^x\right) - \frac{1}{2}t(2\pi)^d\lambda_\varepsilon(0)\right)\right], \tag{5.14}$$

where B^x is a d-dimensional standard Brownian motion independent of W that starts at x and E^B denotes the mathematical expectation with respect to B^x.

Remark 1. Notice that, because u_0 is a signed measure, the composition $u_0(B_t^x)$ is not immediately well defined. The right-hand side of equation (5.14), will be interpreted as follows:

$$\begin{aligned} u^\varepsilon(t,x) = &\int_{\mathbb{R}^d} u_0(d\theta)\boldsymbol{p}_t(x-\theta) \\ &\times \mathrm{E}^{\widehat{B}}\left[\exp\left(\int_{[0,t]\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon(\widehat{B}_{0,t}^{\theta,x}(s)-y)W(ds,dy) - \frac{1}{2}t(2\pi)^d\Lambda_\varepsilon(0)\right)\right], \end{aligned}$$

where $\{\widehat{B}_{0,t}^{\theta,x}(s), s \in [0,t]\}$ denotes a d-dimensional Brownian bridge in the interval $[0,t]$ from θ to x. The above integral is well defined almost surely because on one hand $\int_{\mathbb{R}^d}|u_0|(d\theta)p_t(x-\theta) < \infty$ and moreover, we have

$$\mathrm{E}^W\mathrm{E}^{\widehat{B}}\left[\exp\left(\int_{[0,t]\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon(\widehat{B}_{0,t}^{\theta,x}(s)-y)W(ds,dy)\right)\right] = e^{\frac{1}{2}t(2\pi)^d\lambda_\varepsilon(0)}.$$

Proof of Lemma 5.16. Let $G \in L^2(\Omega,\mathfrak{F},\mathrm{P})$ be such that $G = e^{W(h)-\frac{1}{2}\|h\|_{\mathfrak{H}}^2}$ for some $h \in \mathfrak{H}$. From

(5.14), we obtain

$$\begin{aligned}
&\mathrm{E}\left[Gu^{\varepsilon}(t,x)\right]\\
&=\mathrm{E}^W\left[G\mathrm{E}^B\left[u_0(B_t^x)\exp\left(\int_0^t W^{\varepsilon}\left(ds,B_{t-s}^x\right)-\frac{1}{2}t(2\pi)^d\lambda_{\varepsilon}(0)\right)\right]\right]\\
&=\mathrm{E}^B\left[u_0(B_t^x)\mathrm{E}^W\left[\exp\left(W(h+\boldsymbol{p}_{\varepsilon}(B_{t-\bullet}^x-\star))-\frac{1}{2}\|h\|_{\mathfrak{H}}^2-\frac{1}{2}t(2\pi)^d\lambda_{\varepsilon}(0)\right)\right]\right]\\
&=\mathrm{E}^B\left[u_0(B_t^x)\exp\left(\frac{1}{2}\|h+\boldsymbol{p}_{\varepsilon}(B_{t-\bullet}^x-\star)\|_{\mathfrak{H}}^2-\frac{1}{2}\|h\|_{\mathfrak{H}}^2-\frac{1}{2}t(2\pi)^d\lambda_{\varepsilon}(0)\right)\right]\\
&=\mathrm{E}^B\left[u_0(B_t^x)\exp\left(\left\langle\boldsymbol{p}_{\varepsilon}(B_{t-\bullet}^x-\star),h\right\rangle_{\mathfrak{H}}\right)\right]\\
&=\mathrm{E}^B\left[u_0(B_t^x)\exp\left(\int_0^t\left\langle\boldsymbol{p}_{\varepsilon}(B_{t-s}^x-\star),h(s,\star)\right\rangle_{\mathfrak{H}_0}ds\right)\right].
\end{aligned}$$

Letting $S_{t,x}(h)=\mathrm{E}^W\left[Gu^{\varepsilon}(t,x)\right]$, by the classical Feynmann-Kac's formula, the above calculation shows that $S_{t,x}(h)$ satisfies the classical heat equation with potential $\langle\boldsymbol{p}_{\varepsilon}(x-\star),h(s,\star)\rangle_{\mathfrak{H}_0}$, and initial condition u_0, i.e.

$$\frac{\partial S_{t,x}(h)}{\partial t}=\frac{1}{2}\Delta S_{t,x}(h)+S_{t,x}(h)\left\langle\boldsymbol{p}_{\varepsilon}(x-\star),h(t,\star)\right\rangle_{\mathfrak{H}_0}.$$

As a consequence, we have

$$\begin{aligned}
S_{t,x}(h)&=(\boldsymbol{p}_t*u_0)(x)+\int_0^t\int_{\mathbb{R}^d}\boldsymbol{p}_{t-s}(x-y)S_{s,y}(h)\left\langle\boldsymbol{p}_{\varepsilon}(y-\star),h(s,\star)\right\rangle_{\mathfrak{H}_0}dsdy\\
&=(\boldsymbol{p}_t*u_0)(x)+\int_0^t\int_{\mathbb{R}^d}\boldsymbol{p}_{t-s}(x-y)\mathrm{E}\left[u_{s,y}^{\varepsilon}\left\langle\boldsymbol{p}_{\varepsilon}(y-\star),D_{s,\star}G\right\rangle_{\mathfrak{H}_0}\right]dsdy,
\end{aligned}$$

where we used $DG=hG$. In conclusion, we have proved that

$$\begin{aligned}
\mathrm{E}\left[Gu^{\varepsilon}(t,x)\right]&=(\boldsymbol{p}_t*u_0)(x)\\
&\qquad+\mathrm{E}\left[\left\langle\mathbf{1}_{[0,t]}(\bullet)\int_{\mathbb{R}^d}\boldsymbol{p}_{t-\bullet}(x-y)\boldsymbol{p}_{\varepsilon}(y-\star)u^{\varepsilon}(\bullet,y)dy,DG\right\rangle_{\mathfrak{H}}\right].
\end{aligned}$$

By the fact that the Dalang-Walsh stochastic integral concides with the divergence operator for adapted integrands see Theorem 4.11, we deduce that

$$u^{\varepsilon}(t,x) = (\boldsymbol{p}_t * u_0)(x) \\ + \int_{[0,t]\times\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_{\varepsilon}(y-z)u^{\varepsilon}(s,y)dy \right) W(ds,dz),$$

which implies equation (3.7). □

Next, we will establish the fact that $u^{\varepsilon}(t,x)$ converges to the solution $u(t,x)$ of the stochastic heat equation (5.9) in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ for all $p \geq 1$, and, as a consequence, we derive a Feynman-Kac formula for the moments of the solution u. This type of Feynman-Kac formula has been verifed in the literature under different conditions (see, for instance, [25, Theorem 3.6] for the case where $\Lambda(dx) = \lambda(x)dx$ for a function λ and there is also a correlation in time, or [23] when the noise is white in space and a fractional Brownian motion with Hurst parameter $H > 1/2$ in time) assuming that u_0 is a bounded function. We will give here a detailed proof of this convergence based on the approximation $u^{\varepsilon}(t,x)$. This result and ideas in its proof will be useful in the proof of Theorem 5.18.

Proposition 5.17. Let u^{ε} be the solution to equation (3.7) with an initial condition u_0 satisfying (5.2). Then, for any $k \geq 1$, we have

$$\sup_{\varepsilon>0} \mathrm{E}\left[|u^{\varepsilon}(t,x)|^k \right] < \infty \tag{5.15}$$

and the following convergence holds in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ for any $p \geq 1$,

$$\lim_{\varepsilon\to 0} u^{\varepsilon}(t,x) = u(t,x), \tag{5.16}$$

where u is the solution to the stochastic heat equation (5.9) with initial condition u_0. Furthermore, for any integer $k \geq 2$, the following Feynmann-Kac formula holds,

$$\mathrm{E}\left[u^k(t,x)\right] = \mathrm{E}\left[\prod_{j=1}^{k} u_0(B_t^{j,x}) \exp\left(\sum_{1\leq j<l\leq k} \int_0^t \Lambda(B_s^j - B_s^l) ds\right)\right], \tag{5.17}$$

where $B = \{B^j\}_{j=1,\ldots,k}$ is an independent family of d-dimensional standard Brownian motions and the integrals $\int_0^t \Lambda(B_s^j - B_s^l) ds$ are defined according to Proposition A.4 (ii).

Proof. Set $\Psi_{t,x}^k =: \prod_{j=1}^k u_0(B_t^{j,x})$. Using Lemma A.11, we have

$$\begin{aligned} &\mathrm{E}\left[(u^\varepsilon(t,x))^k\right] \\ =&\mathrm{E}^W \mathrm{E}^B \left[\Psi_{t,x}^k \exp\left(\sum_{j=1}^k \int_0^t W^\varepsilon(ds, B_{t-s}^{j,x}) - \frac{1}{2} t(2\pi)^d \lambda_\varepsilon(0)\right)\right], \end{aligned} \tag{5.18}$$

where $B = \{B^j\}_{j=1,\ldots,k}$ is a family of d-dimensional independent standard Brownian motions independent of W and $B^{j,x} = B^j + x$. Here again the expectation in (5.18) has to be understood as in Remark 1. Changing the order of the expectations, yields

$$\begin{aligned} &\mathrm{E}\left[(u^\varepsilon(t,x))^k\right] \\ &= \mathrm{E}^B\left[\Psi_{t,x}^k \mathrm{E}^W\left[\exp\left(\sum_{j=1}^k \int_0^t \int_{\mathbb{R}^d} \boldsymbol{p}_\varepsilon(B_{t-s}^{j,x} - y) W(ds,dy) - \frac{t(2\pi)^d}{2}\lambda_\varepsilon(0)\right)\right]\right] \\ &= \mathrm{E}\left[\Psi_{t,x}^k \exp\left(\sum_{\substack{j,l=1 \\ j<l}}^k \int_0^t \int_{\mathbb{R}^{2d}} \boldsymbol{p}_\varepsilon(B_{t-s}^{j,x} - y) \boldsymbol{p}_\varepsilon(B_{t-s}^{l,x} - y + y') \Lambda(dy') dy ds\right)\right] \\ &= \mathrm{E}\left[\Psi_{t,x}^k \exp\left(\sum_{1\leq j<l\leq k} \int_0^t \lambda_{2\varepsilon}(B_s^{j,x} - B_s^{l,x}) ds\right)\right]. \end{aligned} \tag{5.19}$$

Integrating with respect to the law of the random vector $(B_t^{1,x}, \ldots, B_t^{k,x})$ whose density is $\theta \mapsto$

$\prod_{j=1}^{k} \boldsymbol{p}_t(x-\theta_j)$, the above expectation can be written as follows.

$$\begin{aligned}\mathrm{E}\left[(u^{\varepsilon}(t,x))^k\right] &= \int_{\mathbb{R}^{kd}} \prod_{j=1}^{k} u_0(d\theta_j)\boldsymbol{p}_t(x-\theta_j) \\ &\quad \times \mathrm{E}\left[\exp\left(\sum_{1\le j<l\le k}\int_0^t \lambda_{2\varepsilon}(\widehat{B}_{0,t}^{j,x,\theta_j}(s)-\widehat{B}_{0,t}^{l,x,\theta_l}(s))ds\right)\right],\end{aligned}$$

where $\left\{\widehat{B}_{0,t}^{j,\theta_j,x}, j=1,\ldots k\right\}$ denotes a family of d-dimensional Brownian bridges in the interval $[0,t]$ from x to θ_j. Now, using the expression (A.4) for Brownian bridges, we can write

$$\begin{aligned}\mathrm{E}\left[(u^{\varepsilon}(t,x))^k\right] &= \int_{\mathbb{R}^{kd}} \prod_{j=1}^{k} u_0(d\theta_j)\boldsymbol{p}_t(x-\theta_j) \\ &\quad \times \mathrm{E}\left[\exp\left(\sum_{1\le j<l\le k}\int_0^t \lambda_{2\varepsilon}\left(\widehat{B}_{0,t}^{j}(s)-\widehat{B}_{0,t}^{l}(s)+\frac{s(\theta_j-\theta_l)}{t}\right)ds\right)\right].\end{aligned} \tag{5.20}$$

Now we can proceed with the proof of the proposition. First, we only need to show (5.15) when k is even. In this case, (5.15) follows from formula (5.20), condition (5.2) and (A.5). Indeed, we have

$$\mathrm{E}\left[(u^{\varepsilon}(t,x))^k\right] \le c_t\left(\int_{\mathbb{R}^d}|u_0|(d\theta)\boldsymbol{p}_t(x-\theta)\right)^k < \infty,$$

where c_t is a finite constant only depending on t. We claim that $u^{\varepsilon}(t,x)$ converges in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ as $\varepsilon\to 0$, for all $p\ge 2$. Indeed,

$$\begin{aligned}\mathrm{E}\left[u^{\varepsilon_1}(t,x)u^{\varepsilon_2}(t,x)\right] &= \int_{\mathbb{R}^{2d}} \prod_{j=1}^{2} u_0(d\theta_j)\boldsymbol{p}_t(x-\theta_j) \\ &\quad \times \mathrm{E}\left[\exp\left(\int_0^t \lambda_{\varepsilon_1+\varepsilon_2}\left(\widehat{B}_{0,t}^{1}(s)-\widehat{B}_{0,t}^{2}(s)+\frac{s(\theta_1-\theta_2)}{t}\right)ds\right)\right]\end{aligned}$$

converges, as $\varepsilon_1,\varepsilon_2$ tend to 0, to

$$\int_{\mathbb{R}^{2d}} \prod_{j=1}^{2} u_0(d\theta_j)\boldsymbol{p}_t(x-\theta_j)\mathrm{E}\left[\exp\left(\int_0^t \Lambda\left(\widehat{B}_{0,t}^{1}(s)-\widehat{B}_{0,t}^{2}(s)+\frac{s(\theta_1-\theta_2)}{t}\right)ds\right)\right]$$

thanks to Proposition A.4. This implies that for any $\varepsilon_k\downarrow 0$, the sequence $u^{\varepsilon_k}(t,x)$ is Cauchy and hence convergent in $L^2(\Omega,\mathfrak{F},\mathrm{P})$ as $k\to\infty$ to some limit $v(t,x)$. The fact that the convergence is in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ follows from (5.19) and Proposition A.4 (i). Taking the limit in (5.19) as ε tends to zero, and using Proposition A.4 (iii), we obtain the Feynman-Kac formula (5.17) for the moments of $v(t,x)$.

It remains to show that $v(t,x)$ coincides with the solution to equation (5.9). By the proof of the Lemma A.11, we know that for any random variable of the form $G=e^{W(h)-\frac{1}{2}\|h\|_{\mathfrak{H}}^2}$ with $h\in\mathfrak{H}$, u^ε satisfies

$$
\begin{aligned}
\mathrm{E}\left[Gu^\varepsilon(t,x)\right] &= (\boldsymbol{p}_t * u_0)(x)\\
&\quad + \mathrm{E}\left[\left\langle \int_{[0,t]\times\mathbb{R}^d} \boldsymbol{p}_{t-s}(x-y)u^\varepsilon(s,y)\boldsymbol{p}_\varepsilon(x-\star)W(ds,dy), D_{s,\star}G\right\rangle_{\mathfrak{H}_0}\right].
\end{aligned}
$$

Now letting $\varepsilon\to 0$, we see that

$$
\mathrm{E}\left[Gv(t,x)\right] = (\boldsymbol{p}_t * u_0)(x) + \mathrm{E}\left[\langle v\boldsymbol{p}_{t-\bullet}(x-\star), DG\rangle_{\mathfrak{H}}\right],
$$

which implies that the process v is also a solution to the equation (5.9), and by uniqueness $v=u$. $\square$

Theorem 5.18. Let u be the unique solution of (5.9) with initial condition u_0 which is a signed Borel measure satisfying (5.2) and $N\geq 1$ an integer. Then for integer $k\geq 2$, we have

$$
\begin{aligned}
\mathrm{E}\left[\left(D^N_{\boldsymbol{r}_N,\boldsymbol{z}_N}u(t,x)\right)^k\right] &= \left[\prod_{m=1}^{N-1}\boldsymbol{p}_{r_{m+1}-r_m}(z_{m+1}-z_m)\right]^k \boldsymbol{p}^k_{t-r_N}(x-z_N)\\
&\quad\times\int_{\mathbb{R}^{kd}}\prod_{j=1}^k u_0(d\theta^j)\prod_{j=1}^k\boldsymbol{p}_{r_1}(z_1-\theta^j)\\
&\quad\times\mathrm{E}\left[\exp\left(\sum_{1\leq j<l\leq k}\int_0^t \Lambda(\widehat{B}^{j,x,z_N,\theta^j}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s)-\widehat{B}^{l,x,z_N,\theta^l}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s))ds\right)\right],
\end{aligned}
$$

for almost all $z_1,\dots,z_N\in\mathbb{R}^d$ and $0<r_1<\cdots<r_N<t$, where $\boldsymbol{t}-\boldsymbol{r}_N=(t-r_1,\dots,t-r_N)$ and $\widehat{B}^{j,x,z_N,\theta^j}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}$, $j=1,\dots,k$ are independent d-dimensional pinned Brownian motions starting from x

with each component pinned at times $t-r_m$ to the points z_m for $1 \le m \le N$, and pinned at θ^j at time t.

Moreover,

$$\mathrm{E}\left[\exp\left(\sum_{1\le j<l\le k}\int_0^t \Lambda(\widehat{B}^{j,x,\boldsymbol{z}_N,\theta^j}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s)-\widehat{B}^{l,x,\boldsymbol{z}_N,\theta^l}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s))ds\right)\right] \le C_{t,k},$$

where $C_{t,k}$ is a constant depending only on t and k.

In the above theorem, taking into account that Λ might be a measure, the composition $\Lambda(\widehat{B}^{j,x,\boldsymbol{z}_N,\theta^j}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s)-\widehat{B}^{l,x,\boldsymbol{z}_N,\theta^l}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s))$ needs to be properly defined as a limit in $L^2(\Omega)$, using an approximation argument, see Proposition A.4, part (ii). When $\Lambda(dx)=\lambda(x)dx$, then this is just an ordinary composition of the density λ with the random variable $\widehat{B}^{j,x,\boldsymbol{z}_N,\theta^j}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s)-\widehat{B}^{l,x,\boldsymbol{z}_N,\theta^l}_{0,\boldsymbol{t}-\boldsymbol{r}_N,t}(s)$.

As a consequence of Theorem 5.18, we deduce the following result.

Corollary 5.19. Under the assumptions and notation of Theorem 5.18, we have

$$\left\|D^N_{\boldsymbol{r}_N,\boldsymbol{z}_N}u(t,x)\right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \le C^{1/p}_{t,p}(\boldsymbol{p}_{r_1}*|u_0|)(z_1)\left(\prod_{m=1}^N \boldsymbol{p}_{r_{m+1}-r_m}(z_{m+1}-z_m)\right), \tag{5.21}$$

where $r_{N+1}=t$, $z_{N+1}=x$.

Corollary 5.20. Under the assumptions of Theorem 5.18,

(i) if $u_0 \equiv 1$, then

$$\begin{aligned} \left\|D_{s,y}u(t,x)\right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} &\le C_{t,p}\boldsymbol{p}_{t-s}(x-y), \text{ and} \\ \left\|D_{r,z}D_{s,y}u(t,x)\right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} &\le C_{t,p}\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_{s-r}(y-z), \end{aligned} \tag{5.22}$$

for all $0<r<s<t$ and $y,z\in\mathbb{R}^d$.

(ii) if $u_0 \equiv \delta_0$, then

$$\begin{aligned} \left\| D_{s,y} u(t,x) \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} &\leq C_{t,p} \boldsymbol{p}_{t-s}(x-y) \boldsymbol{p}_s(y), \text{ and} \\ \left\| D_{r,z} D_{s,y} u(t,x) \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} &\leq C_{t,p} \boldsymbol{p}_{t-s}(x-y) \boldsymbol{p}_{s-r}(y-z) \boldsymbol{p}_r(z), \end{aligned} \tag{5.23}$$

for all $0 < r < s < t$ and $y, z \in \mathbb{R}^d$.

5.2.2 Flat initial condition

We will now investigate the stochastic heat equation (5.1) with flat initial condition $u_0 \equiv 1$. A variation of following result can be found in [45, Proposition 5.1] or [12, Proposition 3.2].

Proposition 5.21. Let u be the mild solution to the stochastic heat equation (5.1) with initial condition $u_0 = 1$, with a noise satisfying the Dalang's condition (5.3). Assume further $g \in C^1(\mathbb{R};\mathbb{R})$ with bounded Lipschitz continuous derivative. Fix $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, then $u(t,x) \in \cap_{p \geq 2} \mathbb{D}^{1,p}$ and for almost all $0 < s < t$, $y \in \mathbb{R}^d$, the derivative $D_{s,y} u(t,x)$ satisfies the following linear stochastic differential equation:

$$\begin{aligned} D_{s,y} u(t,x) &= \boldsymbol{p}_{t-s}(x-y) g(u(s,y)) \\ &\quad + \int_{[s,t] \times \mathbb{R}^d} \boldsymbol{p}_{t-\tau}(x-\xi) g'(u(\tau,\xi)) D_{s,y} u(\tau,\xi) W(d\tau, d\xi). \end{aligned} \tag{5.24}$$

Moreover, for all $0 < s < t \leq T$ and $x, y \in \mathbb{R}^d$, we have

$$\left\| D_{s,y} u(t,x) \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq C_{T,p}\, \boldsymbol{p}_{t-s}(x-y), \tag{5.25}$$

where $C_{T,p}$ is a constant that depends on T, p and g.

The following result is obtained in [31].

Proposition 5.22. Let u be the mild solution to the stochastic heat equation (5.1) with initial condition $u_0 = 1$, in dimension $d = 1$ and the noise is space-time white noise. Assume further

$g \in C^2(\mathbb{R};\mathbb{R})$ with g' bounded and $|g''(x)| \leq C(1+|x|^m)$, for some $m>0$. Fix $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Then $u(t,x) \in \cap_{p\geq 2}\mathbb{D}^{2,p}$ and for almost all $0<r<s<t$, $y,z \in \mathbb{R}$, the second derivative $D_{r,z}D_{s,y}u(t,x)$ satisfies the following linear stochastic differential equation:

$$\begin{aligned} D_{r,z}D_{s,y}u(t,x) = p_{t-s}(x-y)g'(u(s,y))D_{r,z}u(s,y) \\ + \int_{[s,t]\times\mathbb{R}} p_{t-\tau}(x-\xi)g''(u(\tau,\xi))D_{r,z}u(\tau,\xi)D_{s,y}u(\tau,\xi)W(d\tau,d\xi) \\ + \int_{[s,t]\times\mathbb{R}} p_{t-\tau}(x-\xi)g'(u(\tau,\xi))D_{r,z}D_{s,y}u(\tau,\xi)W(d\tau,d\xi). \end{aligned} \tag{5.26}$$

Moreover, for all $0 \leq r < s < t \leq T$ and $x,y,z \in \mathbb{R}$, we have

$$\left\| D_{r,z}D_{s,y}u(t,x) \right\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq C_{T,p}\Phi_{r,z,s,y}(t,x), \tag{5.27}$$

where $C_{T,p}$ is a constant that depends on T, p and g and

$$\begin{aligned} \Phi_{r,z,s,y}(t,x) :=& p_{t-s}(x-y) \\ &\times \left(p_{s-r}(y-z) + \frac{p_{t-r}(z-y)+p_{t-r}(z-x)+\mathbf{1}_{[|y-x|>|z-y|]}}{(s-r)^{1/4}} \right). \end{aligned} \tag{5.28}$$

Remark 5.23. Note that in higher dimensions this problem is still open for general g and spatially homogeneous noise with a general kernel.

Proof of Proposition 5.22 . We will make use of the Picard iteration scheme which is similar to the one used to prove the existence of the mild solution Theorem 5.10. For any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ we put $u_0(t,x) = 1$, and for $n \in \mathbb{N}$ we inductively define

$$u_{n+1}(t,x) = 1 + \int_{[0,t]\times\mathbb{R}} p_{t-\tau}(x-\xi)g(u^n(\tau,\xi))W(d\tau,d\xi).$$

Then, for any $p \geq 2$, there exists a constant $c_{T,p}$ such that for all $(t,x) \in [0,T] \times \mathbb{R}$

$$\sup_{n\in\mathbb{N}} \|u_n(t,x)\|_p \leq c_{T,p}. \tag{5.29}$$

This result is proved in [42, Theorem 2.4.3] for the case of the stochastic heat equation on $[0,1]$ with Dirichlet boundary conditions and the proof works similarly for the equation on $\mathbb{R}$.

We apply the properties of the divergence operator, namely using (4.9), to get that for almost all $(s,y) \in [0,t] \times \mathbb{R}$ and $x \in \mathbb{R}$,

$$D_{s,y}u_{n+1}(t,x) = p_{t-s}(x-y)g_{n,s,y} + \int_{[s,t]\times\mathbb{R}} p_{t-\tau}(x-\xi)g'_{n,\tau,\xi}D_{s,y}u_n(\tau,\xi)W(d\tau,d\xi), \tag{5.30}$$

and for almost all $s > t$, $D_{s,y}u_{n+1}(t,x) = 0$, where we made use of the notation:

$$g_{n,s,y} := g(u_n(s,y)) \quad \text{and} \quad g'_{n,\tau,\xi} := g'(u_n(\tau,\xi)). \tag{5.31}$$

It has also been proven in [27, Lemma A.1] that there is a constant $c_{T,p}$, depending on T and p, such that for almost all $(s,y) \in [0,t] \times \mathbb{R}$ and for all $(t,x) \in [0,T] \times \mathbb{R}$,

$$\sup_{n\in\mathbb{N}} \|D_{s,y}u_n(t,x)\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq c_{T,p}p_{t-s}(x-y). \tag{5.32}$$

Once again using (4.9) and (5.30) together with the Leibniz rule for derivatives, we have, for almost every r,z such that $0 < r < s < t$ and $z \in \mathbb{R}$,

$$\begin{aligned} D_{r,z}D_{s,y}u_{n+1}(t,x) &= p_{t-s}(x-y)g'_{n,s,y}D_{r,z}u_n(s,y) \\ &\quad + \int_{[s,t]\times\mathbb{R}} p_{t-\tau}(x-\xi)g''_{n,\tau,\xi}D_{r,z}u_n(\tau,\xi)D_{s,y}u_n(\tau,\xi)W(d\tau,d\xi) \\ &\quad + \int_{[s,t]\times\mathbb{R}} p_{t-\tau}(x-\xi)g'_{n,\tau,\xi}D_{r,z}D_{s,y}u_n(\tau,\xi)W(d\tau,d\xi), \end{aligned} \tag{5.33}$$

where $g''_{n,\tau,\xi} := g''(u_n(\tau,\xi))$. Applying Burkholder-Davis-Gundy inequality in Theorem 4.10 in

(5.33), the estimate (5.32), hypothesis on g, and the moment estimates (5.29), for any $p \geq 2$ we have for all $(t,x) \in [0,T] \times \mathbb{R}$,

$$
\begin{aligned}
\left\|D_{r,z}D_{s,y}u_{n+1}(t,x)\right\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq\ & C_{T,p}p^2_{t-s}(x-y)p^2_{s-r}(y-z) \\
&+ C_{T,p}\int_s^t \int_{\mathbb{R}} p^2_{t-\tau}(x-\xi)p^2_{\tau-r}(\xi-z)p^2_{\tau-s}(\xi-y)d\xi d\tau \\
&+ C_{T,p}\int_s^t \int_{\mathbb{R}} p^2_{t-\tau}(x-\xi)\|D_{r,z}D_{s,y}u_n(\tau,\xi)\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})}d\xi d\tau, \qquad (5.34)
\end{aligned}
$$

for some constant $C_{T,p} > 0$ which depends on T, p and g. Let J be the measure on $[s,t] \times \mathbb{R}$ defined by

$$
J(d\tau,d\xi) := p^2_{\tau-r}(\xi-z)\delta_{s,y}(d\tau,d\xi) + p^2_{\tau-r}(\xi-z)p^2_{\tau-s}(\xi-y)d\tau d\xi.
$$

Then, we can put the first two summands in (5.34) together and rewrite this inequality as follows:

$$
\begin{aligned}
\left\|D_{r,z}D_{s,y}u_{n+1}(t,x)\right\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq\ & C_{T,p}\int_{[s,t]\times\mathbb{R}} p^2_{t-\tau}(x-\xi)J(d\tau,d\xi) \\
&+ C_{T,p}\int_{[s,t]\times\mathbb{R}} p^2_{t-\tau}(x-\xi)\left\|D_{r,z}D_{s,y}u_n(\tau,\xi)\right\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} d\tau d\xi.
\end{aligned}
$$

After one iteration, this leads to

$$
\begin{aligned}
&\left\|D_{r,z}D_{s,y}u_{n+1}(t,x)\right\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \leq C_{T,p}\int_s^t \int_{\mathbb{R}} p^2_{t-s_1}(x-y_1)J(ds_1,dy_1) \\
&+ C^2_{T,p}\int_s^t \int_{\mathbb{R}}\int_s^{s_1} \int_{\mathbb{R}} p^2_{t-s_1}(x-y_1)p^2_{s_1-s_2}(y_1-y_2)J(ds_2,dy_2)dy_1 ds_1 \\
&+ C^2_{T,p}\int_s^t \int_s^{s_1} \int_{\mathbb{R}^2} p^2_{t-s_1}(x-y_1)p^2_{s_1-s_2}(y_1-y_2)\left\|D_{r,z}D_{s,y}u_{n-1}(s_2,y_2)\right\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} dy_2 dy_1 ds_2 ds_1.
\end{aligned}
$$

If we perform $n-1$ iterations, taking into account that $\left\|D_{r,z}D_{s,y}u_1(s,y)\right\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})}=0$, we obtain

$$\begin{aligned}\left\|D_{r,z}D_{s,y}u_{n+1}(t,x)\right\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} &\le C_{T,p}\int_s^t\int_{\mathbb{R}} p^2_{t-s_1}(x-y_1)J(ds_1,dy_1)\\ &+\sum_{k=1}^{n-1}C^{k+1}_{T,p}\int_s^t\int_{\mathbb{R}}\int_s^{s_1}\int_{\mathbb{R}}\cdots\int_s^{s_k}\int_{\mathbb{R}}p^2_{t-s_1}(x-y_1)p^2_{s_1-s_2}(y_1-y_2)\cdots\\ &\times p^2_{s_k-s_{k+1}}(y_k-y_{k+1})J(ds_{k+1},dy_{k+1})dy_kds_k\cdots dy_1ds_1.\end{aligned}$$

For $0\le r<s<t$, $x,y,z\in\mathbb{R}$, set

$$K^2_{r,z,s,y}(t,x):=\int_s^t\int_{\mathbb{R}}p^2_{t-s_1}(x-y_1)J(ds_1,dy_1). \tag{5.35}$$

For the sake of simplicity, we use $K^2(t,x)$ to denote $K^2_{r,z,s,y}(t,x)$. The identity $p^2_t(x)=\frac{1}{\sqrt{2\pi t}}p_{t/2}(x)$ now implies

$$\begin{aligned}\|D_{r,z}D_{s,y}u_{n+1}(t,x)\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} &\le C_{T,p}K^2(t,x)\\ &+\sum_{k=1}^{n-1}\frac{C^{k+1}_{T,p}}{(2\pi)^{\frac{k+1}{2}}}\int_{s<s_{k+1}<\cdots<s_2<s_1<t}ds_1\cdots ds_k\int_{\mathbb{R}^{k+1}}dy_1\cdots dy_k\\ &\times\left[(t-s_1)(s_1-s_2)\cdots(s_k-s_{k+1})\right]^{-\frac{1}{2}}\\ &\times p_{\frac{t-s_1}{2}}(x-y_1)p_{\frac{s_1-s_2}{2}}(y_1-y_2)\cdots p_{\frac{s_k-s_{k+1}}{2}}(y_k-y_{k+1})J(ds_{k+1},dy_{k+1}).\end{aligned}$$

Integrating in the variables $y_1,\dots,y_k$ and using the semigroup property of the heat kernel yields

$$
\begin{aligned}
\|D_{r,z}D_{s,y}u_{n+1}(t,x)\|^2_{L^p(\Omega,\mathfrak{F},\mathrm{P})} &\le C_{T,p}K^2(t,x) + \sum_{k=1}^{n-1} \frac{C_{T,p}^{k+1}}{(2\pi)^{\frac{k+1}{2}}} \int_{s<s_{k+1}<\cdots<s_2<s_1<t} ds_k \cdots ds_1 \\
&\quad \times [(t-s_1)(s_1-s_2)\cdots(s_k-s_{k+1})]^{-\frac{1}{2}} \int_{\mathbb{R}} p_{\frac{t-s_{k+1}}{2}}(x-y_{k+1}) J(ds_{k+1},dy_{k+1}) \\
&= C_{T,p}K^2(t,x) + \sum_{k=1}^{n-1} \frac{C_{T,p}^{k+1}}{(2\pi)^{\frac{k+1}{2}}} \int_{0<r_k<\cdots<r_2<r_1<1} dr_k \cdots dr_1 \\
&\quad \times [(1-r_1)(r_1-r_2)\cdots r_k]^{-\frac{1}{2}} \int_{\mathbb{R}} \int_s^t (t-\tau)^{\frac{k}{2}} p_{\frac{t-\tau}{2}}(x-\xi) J(d\tau,d\xi) \\
&= C_{T,p}K^2(t,x) + \sum_{k=1}^{n-1} \frac{\Gamma(1/2)^k C^{k+1}}{(2\pi)^{\frac{k}{2}}\Gamma(k/2)} \int_{\mathbb{R}} \int_s^t (t-\tau)^{\frac{k+1}{2}} p^2_{t-\tau}(x-\xi) J(\tau,d\xi) \\
&\le CK^2(t,x) + \sum_{k=1}^{n-1} \frac{\Gamma(1/2)^k C_{T,p}^{k+1} T^{\frac{k+1}{2}}}{(2\pi)^{\frac{k}{2}}\Gamma(k/2)} \int_{\mathbb{R}} \int_s^t p^2_{t-\tau}(x-\xi) J(d\tau,d\xi) \\
&\le \left(C_{T,p} + \sum_{k=1}^{\infty} \frac{\Gamma(1/2)^k C_{T,p}^{k+1} T^{\frac{k+1}{2}}}{(2\pi)^{\frac{k}{2}}\Gamma(k/2)} \right) K^2(t,x) =: \widetilde{C}^2_{T,p} K^2(t,x).
\end{aligned}
$$

Using Lemma A.6, we arrive at the upper-bound

$$
\sup_{n\in\mathbb{N}} \|D_{r,z}D_{s,y}u_n(t,x)\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \le \widetilde{C}_{T,p}\Phi_{r,z,s,y}(t,x).
$$

As a consequence, applying Minkowski's inequality and then using Lemma A.7 we can write

$$
\begin{aligned}
\sup_{n\in\mathbb{N}} \mathrm{E}\left[\left\|D^2 u_n(t,x)\right\|^p_{\mathfrak{H}\otimes\mathfrak{H}} \right] &\le \sup_{n\in\mathbb{N}} \left(\int_{[0,t]^2} \int_{\mathbb{R}^2} \|D_{r,z}D_{s,y}u_n(t,x)\|^2_p dydzdrds \right)^{\frac{p}{2}} \\
&\le \widetilde{C}^p_{T,p} \left(2\int_0^t \int_0^s \int_{\mathbb{R}^2} \Phi^2_{r,z,s,y}(t,x) dzdydrds \right)^{\frac{p}{2}} < \infty.
\end{aligned}
$$

Since $u_n(t,x)$ converges in $L^p(\Omega,\mathfrak{F},\mathrm{P})$ to $u(t,x)$ for all $p \ge 2$, using Lemma 2.26 we deduce that $u(t,x) \in \cap_{p\ge 2}\mathbb{D}^{2,p}$. Following the arguments in the proof of [16, Theorem 6.4] we deduce

$$
\|D_{r,z}D_{s,y}u(t,x)\|_{L^p(\Omega,\mathfrak{F},\mathrm{P})} \le \widetilde{C}_{T,p}\Phi_{r,z,s,y}(t,x).
$$

□

Chapter 6

Study of spatial averages

In this chapter, we investigate the recent results on the study of spatial averages of the solution to stochastic heat equation. For the sake of simplicity, we focus on the case where the equation is governed by a space-time white noise in dimension 1 and the initial condition is either constant or dirac mass at 0. This chapter is based on the papers Chen, Khoshnevisan, Nualart, and Pu [16, 15], Huang, Nualart, and Viitasaari [27], Kuzgun and Nualart [31].

6.1 Flat initial condition in SHE

Let the random field $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the mild solution to the stochastic heat equation (5.1) in dimension 1 with initial condition $u_0 \equiv 1$ and a space-time white noise W. Set $Q_R := [-R,R]$, and define for $0 < s < t$ and $y \in \mathbb{R}$

$$\phi_{R,t}(s,y) := \frac{1}{\sigma_{R,t}} \int_{Q_R} p_{t-s}(x-y)dx. \tag{6.1}$$

Fix $R > 0$ and consider the corresponding centered and normalized spatial averages defined by

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left(\int_{Q_R} u(t,x)dx - 2R \right), \text{ where } \sigma^2_{R,t} := \mathrm{Var}\left[\int_{-R}^{R} u(t,x)dx \right]. \tag{6.2}$$

For any fixed $t > 0$, the random variable $F_{R,t}$ defined in (6.2) is given by

$$\begin{aligned} F_{R,t} &= \frac{1}{\sigma_{R,t}} \left(\int_{Q_R} \int_{[0,t]\times\mathbb{R}} p_{t-s}(x-y)g(u(s,y))W(ds,dy)\,dx \right) \\ &= \int_{[0,t]\times\mathbb{R}} \frac{1}{\sigma_{R,t}} \left(\int_{Q_R} p_{t-s}(x-y)g(u(s,y))dx \right) W(ds,dy), \end{aligned}$$

where we recall that $Q_R = [-R,R]$. So, taking into account that the Itô-Walsh stochastic integral coincides with the divergence operator for adapted integrands, we obtain the representation

$$F_{R,t} = \delta(v_{R,t}),$$

where

$$v_{R,t}(s,y) = \mathbf{1}_{[0,t]}(s) \frac{1}{\sigma_{R,t}} \int_{Q_R} p_{t-s}(x-y)g(u(s,y))dx. \tag{6.3}$$

Lemma 6.1. Let $F_{R,t}$ and $\sigma_{R,t}$ be as defined in (6.2) and set $\eta(s) = \mathrm{E}\left[(g(u(s,y)))^2\right]$ which doesn't depend on y by stationarity of u. Then, for any $s,t \geq 0$,

$$\lim_{R\to\infty} \frac{1}{R} \mathrm{Cov}\left[\int_{-R}^{R} u(t,x)dx - 2R, \int_{-R}^{R} u(t,x)dx - 2R\right] = 2\int_0^{s\wedge t} \eta(\tau)d\tau.$$

In particular,

$$\lim_{R\to\infty} \frac{\sigma_{R,t}^2}{R} = 2\int_0^t \eta(\tau)d\tau.$$

Proof. Using the mild formulation (5.6) of u followed by Itô-Walsh isometry (4.6) and semigroup property of heat kernel, we have

$$\begin{aligned} \mathrm{E}[u(t,x)u(s,y)] &= 1 + \int_0^{s\wedge t} \int_{\mathbb{R}} p_{t-\tau}(x-\xi)p_{s-\tau}(y-\xi)\mathrm{E}\left[(g(u(\tau,\xi)))^2\right] d\xi d\tau \\ &= 1 + \int_0^{t\wedge s} \eta(\tau) \int_{\mathbb{R}} p_{t-\tau}(x-\xi)p_{s-\tau}(y-\xi)d\xi d\tau \\ &= 1 + \int_0^{t\wedge s} \eta(s)p_{t+s-2\tau}(x-y)d\tau. \end{aligned}$$

Using the fact that

$$\mathrm{E}\left[\int_{-R}^{R} u(t,x)dx\right] = 2R,$$

we obtain

$$\begin{aligned}
\mathrm{Cov}\left[\int_{-R}^{R} u(t,x)dx - 2R, \int_{-R}^{R} u(t,x)dx - 2R\right] &= \int_0^{s\wedge t} \eta(\tau)\int_{-R}^{R}\int_{-R}^{R} p_{t+s-2\tau}(x-y)dxdyd\tau \\
&= 2\int_0^{s\wedge t} \eta(\tau)\int_0^{2R} p_{t+s-2\tau}(y)(2R-y)dyd\tau.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\lim_{R\to\infty} \mathrm{Cov}\left[\int_{-R}^{R} u(t,x)dx - 2R, \int_{-R}^{R} u(t,x)dx - 2R\right] &= \lim_{R\to\infty} 2\int_0^{s\wedge t} \eta(\tau)\int_0^{2R} p_{t+s-2\tau}(y)(2-\frac{y}{R})dyd\tau \\
&= 2\int_0^{t\wedge s} \eta(\tau)d\tau
\end{aligned}$$

which completes the proof of our claim. □

Theorem 6.2. For every $t > 0$, there exist $C_t = C(t) > 0$ such that for all $R \geq 1$,

$$d_{\mathrm{TV}}(F_{R,t}, N) \leq C_t \frac{1}{\sqrt{R}}. \tag{6.4}$$

Proof. By (3.4), and (6.3), we know

$$d_{\mathrm{TV}}(F_{R,t}, N) \leq 2\sqrt{\mathrm{Var}\left[\langle DF_{R,t}, v_{R,t}\rangle\right]}.$$

Now consider

$$D_{s,y}F_{R,t} = \frac{1}{\sigma_{R,t}}\int_{-R}^{R} D_{s,y}u(t,x)dx. \tag{6.5}$$

and

$$\langle DF_{R,t}, v_{R,t}\rangle_{\mathfrak{H}} = \frac{1}{\sigma_{R,t}^2}\int_0^t\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t-s}(x-y)g(u(s,y))D_{s,y}u(t,x')dxdx'dyds.$$

Using (5.24) we get

$$\begin{aligned}\langle DF_{R,t}, v_{R,t}\rangle_{\mathfrak{H}} =& \frac{1}{\sigma_{R,t}^2}\int_0^t\int_{\mathbb{R}}\left(\int_{-R}^{R} p_{t-s}(x-y)dx\right)^2 g^2(u(s,y))dyds \\ &+\frac{1}{\sigma_{R,t}^2}\int_0^t\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t-s}(x-y)g(u(s,y)) \\ &\left(\int_{[s,t]\times\mathbb{R}} p_{t-\tau}(x'-\xi)g'_{\tau,\xi}D_{s,y}u(\tau,\xi)W(d\tau,d\xi)\right)dxdx'dyds.\end{aligned} \tag{6.6}$$

Now, using Lemma A.12, we can estimate $\sqrt{\mathrm{Var}\left[\langle DF_{R,t}, v_{R,t}\rangle_{\mathfrak{H}}\right]}$ as follows:

$$\sqrt{\mathrm{Var}\left[\langle DF_{R,t}, v_{R,t}\rangle_{\mathfrak{H}}\right]} \le \int_0^t a(s)+b(s)ds \tag{6.7}$$

where

$$a(s) := \frac{1}{\sigma_{R,t}^2}\sqrt{\mathrm{Var}\left[\int_{\mathbb{R}}\left(\int_{-R}^{R} p_{t-s}(x-y)dx\right)^2 g^2(u(s,y))dy\right]} \tag{6.8}$$

$$b(s) := \frac{1}{\sigma_{R,t}^2}\sqrt{\mathrm{Var}\left[\int_{\mathbb{R}}\int_{-R}^{R}\int_{-R}^{R} p_{t-s}(x-y)g_{s,y}\left(\int_{[s,t]\times\mathbb{R}} p_{t-\tau}(x'-\xi)g'_{\tau,\xi}D_{s,y}u(\tau,\xi)W(d\tau,d\xi)\right)dxdx'dy\right]}. \tag{6.9}$$

Now we will estimate both of these terms in two steps.

Estimate for $a(s)$: Note that

$$a(s) = \frac{1}{\sigma_{R,t}^2}\sqrt{\int_{\mathbb{R}^2}\left(\int_{-R}^{R} p_{t-s}(x-y)dx\right)^2\int_{\mathbb{R}^2}\left(\int_{-R}^{R} p_{t-s}(x'-y')dx\right)^2 \mathrm{Cov}\left[g^2(u(s,y)), g^2(u(s,y'))\right]dydy'}. \tag{6.10}$$

Let us start by estimating $\mathrm{Cov}\left[g^2(u(s,y)), g^2(u(s,y'))\right]$. Using Theorem 4.12, we write

$$g^2(u(s,y)) = \mathrm{E}\left[g^2(u(s,y))\right] + \int_{[0,s]\times\mathbb{R}} \mathrm{E}\left[D_{\tau,\xi}(g^2(u(s,y)))\middle|\mathfrak{F}_\tau\right] W(d\tau, d\xi).$$

Using this representation, and Itô-Walsh isometry Proposition 4.8, we see

$$\mathrm{Cov}\left[g^2(u(s,y)), g^2(u(s,y'))\right] = \int_0^s \int_{\mathbb{R}} \mathrm{E}\left[\mathrm{E}\left[D_{\tau,\xi}(g^2(u(s,y)))\middle|\mathfrak{F}_\tau\right] \mathrm{E}\left[D_{\tau,\xi}(g^2(u(s,y')))\middle|\mathfrak{F}_\tau\right]\right] d\xi d\tau. \tag{6.11}$$

Applying the chain rule in Proposition Proposition 2.24, we have

$$D_{\tau,\xi}(g^2(u(s,y))) = 2g_{s,y}g'_{s,y}D_{\tau,\xi}u(s,y).$$

Now let Lip_g be the Lipschitz constant of g and

$$K_p(t) := \sup_{(s,y)\in[0,t]\times\mathbb{R}} \|g(u(s,y))\|_p < \infty \tag{6.12}$$

since moments of u are finite for fixed $[0,t]$. Using this notation, together with contractivity of conditional expectation and Hölder's inequality, we obtain

$$\left\|\mathrm{E}\left[D_{\tau,\xi}(g^2(u,sy))\middle|\mathfrak{F}_\tau\right]\right\|_2 = \left\|\mathrm{E}\left[2g_{s,y}g'_{s,y}D_{\tau,\xi}u(s,y)\middle|\mathfrak{F}_\tau\right]\right\|_2 \leq 2K_4(t)\mathrm{Lip}_g \left\|D_{\tau,\xi}u(s,y)\right\|_4.$$

Now, using the above estimate in (6.11) together with (5.25) and Hölder's inequality, we see

$$\begin{aligned}\mathrm{Cov}\left[g^2(u(s,y)), g^2(u(s,y'))\right] &\leq 4K_4^2(t)\mathrm{Lip}_g^2 \int_0^s \int_{\mathbb{R}} \left\|D_{\tau,\xi}u(s,y)\right\|_4 \left\|D_{\tau,\xi}u(s,y')\right\|_4 d\xi d\tau \\ &\leq 4K_4^2(t)\mathrm{Lip}_g^2 \int_0^s \int_{\mathbb{R}} p_{s-\tau}(\xi-y)p_{s-\tau}(\xi-y')d\xi d\tau \\ &= 4K_4^2(t)\mathrm{Lip}_g^2 \int_0^s p_{2s-2\tau}(y-y')d\tau.\end{aligned}$$

Using this bound, we can now estimate (6.10) as follows:

$$
\begin{aligned}
a(s) \leq & \frac{2K_4(t)\mathrm{Lip}_g}{\sigma_{R,t}^2}\Big(\int_0^s \int_{\mathbb{R}^2}\int_{[-R,R]^4} p_{t-s}(x-y)p_{t-s}(\tilde{x}-y)p_{t-s}(x'-y')p_{t-s}(\tilde{x}'-y') \\
& p_{2s-2\tau}(y-y')dx'dxd\tilde{x}'d\tilde{x}dydy'd\tau\Big)^{1/2}.
\end{aligned}
$$

Integrating in $\tilde{x},\tilde{x}'$ over whole line and using Lemma 6.1, and then y,y' using semigroup property, we get

$$
a(s) \leq \frac{C_t}{R}\left(\int_0^s \int_{[-R,R]^2} p_{2t-2\tau}(x-x')dx'dxd\tau\right)^{1/2}. \tag{6.13}
$$

Finally, integrating x over $\mathbb{R}$, we obtain

$$
\begin{aligned}
a(s) &\leq \frac{C_t}{R}\left(\int_0^s \int_{[-R,R]} dxd\tau\right)^{1/2} && (6.14)\\
&\leq \frac{C_t}{\sqrt{R}}. && (6.15)
\end{aligned}
$$

Estimate for $b(s)$: Using Burkholder-David-Gundy inequality in Theorem 4.10, we have

$$
\begin{aligned}
b(s) \leq \frac{1}{\sigma_{R,t}^2}\Big(\int_s^t \int_{\mathbb{R}^3}\int_{[-R,R]^4} & p_{t-s}(x-y)p_{t-s}(x'-y')p_{t-\tau}(\tilde{x}-\xi)p_{t-\tau}(\tilde{x}'-\xi) \\
& \mathrm{E}\left[g_{s,y}g_{s,y'}g'^2_{\tau,\xi}D_{s,y}u(\tau,\xi)D_{s,y'}u(\tau,\xi)\right]dxdx'd\tilde{x}d\tilde{x}'dydy'd\xi d\tau\Big)^{1/2}
\end{aligned}
$$

Recalling the notation (6.12) and the estimate (5.24), and applying Hölder's inequality, we have

$$
\mathrm{E}\left[g_{s,y}g_{s,y'}g'^2_{\tau,\xi}D_{s,y}u(\tau,\xi)D_{s,y'}u(\tau,\xi)\right] \leq K_4^2(t)\mathrm{Lip}_g^2 p_{\tau-s}(\xi-y)p_{\tau-s}(\xi-y').
$$

Using this and Lemma 6.1 in $b(s)$, we get

$$b(s) \leq \frac{C_t}{R}\bigg(\int_s^t \int_{\mathbb{R}^3}\int_{[-R,R]^4} p_{t-s}(x-y)p_{t-s}(x'-y')p_{t-\tau}(\tilde{x}-\xi)p_{t-\tau}(\tilde{x}'-\xi)$$
$$p_{\tau-s}(\xi-y)p_{\tau-s}(\xi-y')dxdx'd\tilde{x}d\tilde{x}'dydy'd\xi d\tau\bigg)^{1/2}.$$

Integrating over $\tilde{x},\tilde{x}'$ over $\mathbb{R}$ and integrating y and y' using semigroup property, we see

$$\begin{aligned} b(s) &\leq \frac{C_t}{R}\left(\int_s^t \int_{[-R,R]^2}\int_{\mathbb{R}} p_{t+\tau-2s}(x-\xi)p_{t+\tau-2s}(x'-\xi)d\xi dxdx'd\tau\right)^{1/2} \\ &\leq \frac{C_t}{R}\left(\int_s^t \int_{[-R,R]^2}\int_{\mathbb{R}} p_{2t+2\tau-4s}(x-x')dxdx'd\tau\right)^{1/2} \end{aligned}$$

where we used semigroup property integrating in ξ. Finally, integrating x in all $\mathbb{R}$, we get

$$b(s) \leq \frac{C_t}{\sqrt{R}}.$$

Putting these two cases together in (6.7), we obtain

$$\sqrt{\operatorname{Var}\left[\langle DF_{R,t}, v_{R,t}\rangle_{\mathfrak{H}}\right]} \leq \frac{C_t}{\sqrt{R}}, \tag{6.16}$$

which completes our proof.

□

Proposition 6.3. Let u be the solution to the integral equation (5.6) and assume that g is Lipschitz. Fix $p \geq 2$, $t > 0$ and assume that there exists $q > 5p$ such that $\mathrm{E}\left[|g(u(t,0))|^{-2q}\right] < \infty$. Then, there exists $R_0 > 0$ such that

$$\sup_{R\geq R_0} \mathrm{E}\left[\left|D_{v_{R,t}}F_{R,t}\right|^{-p}\right] < \infty. \tag{6.17}$$

Proof. Consider the Malliavin derivative of $F_{R,t}$ given by

$$D_{r,z}F_{R,t} = \frac{1}{\sigma_{R,t}}\int_{Q_R} dx D_{r,z}u(t,x).$$

From (6.3) and (6.2), we can write

$$D_{v_{R,t}}F_{R,t} = \int_0^t \int_{\mathbb{R}} v_{R,t}(r,z) D_{r,z}F_{R,t}\,dzdr$$
$$= \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} \int_0^t \int_{\mathbb{R}} p_{t-r}(x_1 - z) g(u(r,z)) D_{r,z}u(t,x_2)\,dzdrdx_1dx_2$$
$$= \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} \int_0^t \int_{\mathbb{R}} p_{t-r}(x_1 - z) g^2(u(r,z)) \Psi^{r,z}(t,x_2)\,dzdrdx_1dx_2, \qquad (6.18)$$

with the notation

$$\Psi^{r,z}(t,x) = \frac{D_{r,z}u(t,x)}{g(u(r,z))},$$

for any $r < t$. Notice that $g(u(r,z)) \neq 0$ almost surely because $\mathrm{E}\left[|g(u(r,z))|^{-2q}\right] < \infty$ due to our hypothesis and the stationarity of the process $\{u(r,z) : z \in \mathbb{R}\}$.

We claim that

$$\Psi^{r,z}(t,x) \geq 0. \qquad (6.19)$$

Indeed, from equation (5.24), it follows that$\{\Psi^{r,z}(t,x) : (t,x) \in [r,\infty) \times \mathbb{R}\}$ satisfies:

$$\Psi^{r,z}(t,x) = p_{t-r}(x-z) + \int_{[r,t]\times\mathbb{R}} p_{t-s}(x-y) g'(u(s,y)) \Psi^{r,z}(s,y) W(ds,dy).$$

That means,$\Psi^{r,z}(t,x)$ solves the heat equation

$$\frac{\partial \Psi^{r,z}}{\partial t} = \frac{1}{2}\frac{\partial^2 \Psi^{r,z}}{\partial x^2} + g'(u)\Psi^{r,z}\dot{W}, \qquad x \in \mathbb{R},\ t \in [r,\infty),$$

with initial condition $\Psi^{r,z}(t,x)\big|_{t=r} = \delta_z(x)$ and, in particular, $\Psi^{r,z}(t,x)$ is nonnegative.

As a consequence, from (6.18) and (6.19) it follows that $D_{v_{R,t}}F_{R,t} \geq 0$ and we can write

$$D_{v_{R,t}}F_{R,t} \geq \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} \int_{t-\varepsilon^\alpha}^t \int_{\mathbb{R}} p_{t-r}(x_1 - z) g(u(r,z)) D_{r,z}u(t,x_2)\,dzdrdx_1dx_2,$$

for any $\varepsilon < t$ and $\alpha < 1$. Set $t_{\varepsilon^\alpha} := t - \varepsilon^\alpha$. Using this estimate, we get

$$\mathrm{P}\left(D_{v_{R,t}} F_{R,t} < \varepsilon\right) \leq \mathrm{P}\left(\frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} \int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} p_{t-r}(x_1 - z) g(u(r,z)) D_{r,z} u(t, x_2) dz dr dx_1 dx_2 < \varepsilon\right).$$

With the notation (6.1), using (5.24) we obtain

$$\begin{aligned}
&\frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} \int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} p_{t-r}(x_1 - z) g(u(r,z)) D_{r,z} u(t, x_2) dz dr dx_1 dx_2 \\
&= \int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \phi_{R,t}^2(r,z) g^2(u(r,z)) dz dr \\
&+ \int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \left(\int_{[r,t]\times\mathbb{R}} \phi_{R,t}(s,y) \phi_{R,t}(r,z) g'(u(s,y)) D_{r,z} u(s,y) W(ds,dy) \right) g(u(r,z) dz dr \\
&=: I_1 + I_2.
\end{aligned}$$

From

$$\begin{aligned}
\mathrm{P}\left(I_1 + I_2 < \varepsilon\right) &\leq \mathrm{P}\left(I_1 < 2\varepsilon\right) + \mathrm{P}\left(I_1 + I_2 < \varepsilon, I_1 \geq 2\varepsilon\right) \\
&\leq \mathrm{P}\left(I_1 < 2\varepsilon\right) + \mathrm{P}\left(|I_2| > \varepsilon\right),
\end{aligned} \tag{6.20}$$

we have

$$\begin{aligned}
&\mathrm{P}\left(\frac{1}{\sigma_R^2} \int_{Q_R^2} \int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} p_{t-r}(x_1 - z) g(u(r,z)) D_{r,z} u(t, x_2) dz dr dx_1 dx_2 < \varepsilon\right) \\
&\leq \mathrm{P}\left(I_1 < 2\varepsilon\right) + \mathrm{P}\left(|I_2| > \varepsilon\right).
\end{aligned}$$

We shall next estimate these probabilities, starting with the first one:

$$\begin{aligned}
\mathrm{P}\left(I_1 < 2\varepsilon\right) &= \mathrm{P}\left(\int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \phi_{R,t}^2(r,z) g^2(u(r,z)) dz dr < \varepsilon\right) \\
&= \mathrm{P}\left(\int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \phi_{R,t}^2(r,z) \Big(g(u(r,z)) - g(u(t,z)) + g(u(t,z)) \Big)^2 dz dr < 2\varepsilon\right).
\end{aligned}$$

Using the inequality $(a+b)^2 \geq a^2/2 - b^2$ for $a,b \in \mathbb{R}$, and an estimate similar to (6.20), we get

$$\begin{aligned}
&\mathrm{P}\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)\left(g(u(r,z))-g(u(t,z)+g(u(t,z))\right)^2dzdr<2\varepsilon\right)\\
&\leq \mathrm{P}\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)\left(g(u(t,z))\right)^2dzdr<6\varepsilon\right)\\
&\quad+\mathrm{P}\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)\left(g(u(r,z))-g(u(t,z))\right)^2dzdr>\varepsilon\right)\\
&=: K_1+K_2. \qquad (6.21)
\end{aligned}$$

For the term K_1 in (6.21), by Chebyshev's inequality, for $q>5p$ we obtain

$$\begin{aligned}
K_1 &= \mathrm{P}\left(\left[\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)g^2(u(t,z))dzdr\right]^{-1}>\frac{1}{6\varepsilon}\right)\\
&\leq (6\varepsilon)^q\mathrm{E}\left[\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)g^2(u(t,z))dzdr\right)^{-q}\right]. \qquad (6.22)
\end{aligned}$$

Set

$$m(\varepsilon,R) := \int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)dzdr.$$

Then, taking into account that the function $x \to x^{-q}$ is convex and applying Jensen's inequality, we can write

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)g^2(u(t,z))dzdr\right)^{-q}\right]\\
&= m(\varepsilon,R)^{-q}\mathrm{E}\left[\left(\frac{1}{m(\varepsilon,R)}\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)g^2(u(t,z))dzdr\right)^{-q}\right]\\
&\leq m(\varepsilon,R)^{-q-1}\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\phi_{R,t}^2(r,z)\mathrm{E}\left[|g(u(t,z))|^{-2q}\right]drdz. \qquad (6.23)
\end{aligned}$$

Since the solution is stationary in space, the factor $C_t := \mathrm{E}\left[|g(u(t,z))|^{-2q}\right]$ does not depend on z and we assume it is finite. Therefore, from (6.22) and (6.23), we get

$$K_1 \leq C_t(6\varepsilon)^q m(\varepsilon,R)^{-q} \qquad (6.24)$$

for some constant $C_t > 0$. Moreover,

$$
\begin{aligned}
m(\varepsilon,R) &= \frac{1}{\sigma_{R,t}^2}\int_0^{\varepsilon^\alpha}\int_{-R}^{R}\int_{-R}^{R} p_{2s}(x_1-x_2)dx_1dx_2ds \\
&\geq \frac{\sqrt{2}R}{\sigma_{R,t}^2}\int_0^{\varepsilon^\alpha}\int_{-R/\sqrt{2}}^{R/\sqrt{2}} p_{2s}(y)dyds.
\end{aligned}
$$

Then, assuming $\varepsilon \leq 1$ and $R \geq R_0$, we obtain

$$
m(\varepsilon,R) \geq \frac{\sqrt{2}R}{\sigma_{R,t}^2}\int_0^{\varepsilon^\alpha}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} p_2(y)dyds \geq C_t\varepsilon^\alpha, \tag{6.25}
$$

where in the last inequality we have used Lemma 6.1. Hence, from (6.24) and (6.25), we have

$$
K_1 \leq C_t \varepsilon^{q(1-\alpha)}. \tag{6.26}
$$

In order to estimate the term K_2 in (6.21), we use Chebyschev's inequality followed by Minkowski's inequality, as follows:

$$
\begin{aligned}
K_2 &\leq \varepsilon^{-q}\mathrm{E}\left[\left(\int_{t_{\varepsilon^\alpha}}^{t}\int_{\mathbb{R}} \phi_{R,t}^2(r,z)(g(u(r,z))-g(u(t,z)))^2dzdr\right)^q\right] \\
&\leq \varepsilon^{-q}\left(\int_{t_{\varepsilon^\alpha}}^{t}\int_{\mathbb{R}} \phi_{R,t}^2(r,z)\left(\mathrm{E}\left[|g(u(r,z))-g(u(t,z))|^{2q}\right]\right)^{1/q}dzdr\right)^q.
\end{aligned} \tag{6.27}
$$

The Lipschitz continuity of g and the $1/4$-Hölder continuity of the solution $u(t,x)$ in $L^{2q}(\Omega)$ allow us to write for any $r \in [t_\alpha, t]$

$$
\begin{aligned}
\|g(u(r,z))-g(u(t,z))\|_{2q} &\leq \mathrm{Lip}_g\|u(r,z)-u(t,z)\|_{2q} \\
&\leq C_t\mathrm{Lip}_g|t-r|^{1/4} \leq C_t\mathrm{Lip}_g\varepsilon^{\alpha/4}.
\end{aligned} \tag{6.28}
$$

On the other hand, from (6.25) we have, for $R \geq R_0$,

$$\int_{t_{\varepsilon^\alpha}}^{t} \int_{\mathbb{R}} \phi_{R,t}^2(r,z) dz dr = \frac{1}{\sigma_{R,t}^2} \int_0^{\varepsilon^\alpha} \int_{Q_R^2} \int_{\mathbb{R}} p_r(x_1 - z) p_r(z - x_2) dz dx_1 dx_2 dr$$
$$\leq \frac{1}{\sigma_{R,t}^2} \int_0^{\varepsilon^\alpha} \int_{Q_R^2} \int_{\mathbb{R}} p_{2r}(x_1 - x_2) dx_1 dx_2 dr \leq \frac{2R}{\sigma_{R,t}^2} \varepsilon^\alpha \leq C_t \varepsilon^\alpha. \tag{6.29}$$

Substituting (6.28) and (6.29) into (6.27), yields

$$K_2 \leq C_t \varepsilon^{(\frac{3\alpha}{2} - 1)q}. \tag{6.30}$$

We are left to estimate the following probability:

$$K_3 := \mathrm{P}\left(|I_2| > \varepsilon\right).$$

Using Fubini's theorem and Chebyschev's inequality, we have

$$K_3 \leq \frac{1}{\varepsilon^q} \mathrm{E}\left[\left| \int_{[t_{\varepsilon^\alpha}, t] \times \mathbb{R}} \int_{\mathbb{R}} \int_{t_{\varepsilon^\alpha}}^{s} \phi_{R,t}(r,z) \phi_{R,t}(s,y) g'_{s,y} D_{r,z} u(s,y) g_{r,z} dr dz W(ds,dy) \right|^q \right].$$

Then, applying Burkholder-Davis-Gundy inequality in Theorem 4.10, followed by Minkowski's inequality, we get

$$
\begin{aligned}
K_3 &\le \frac{C_q}{\varepsilon^q}\mathrm{E}\left[\left|\int_{t_\varepsilon\alpha}^t\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{t_\varepsilon\alpha}^s \phi_{R,t}(r,z)\phi_{R,t}(s,y)g'_{s,y}D_{r,z}u(s,y)g_{r,z}drdz\right)^2 dsdy\right|^{\frac{q}{2}}\right] \\
&= \frac{C_q}{\varepsilon^q}\mathrm{E}\left[\left|\int_{t_\varepsilon\alpha}^t\int_{t_\varepsilon\alpha}^s\int_{t_\varepsilon\alpha}^s\int_{\mathbb{R}^3}\phi_{R,t}(r_1,z_1)\phi_{R,t}(r_2,z_2)\phi_{R,t}^2(s,y)\right.\right. \\
&\qquad \left.\left.\times X_{r_1,z_1,r_2,z_2}(s,y)dz_1dz_2dydr_1dr_2ds\right|^{\frac{q}{2}}\right] \\
&\le \frac{C_q}{\varepsilon^q}\left(\int_{t_\varepsilon\alpha}^t\int_{t_\varepsilon\alpha}^s\int_{t_\varepsilon\alpha}^s\int_{\mathbb{R}^3}\phi_{R,t}(r_1,z_1)\phi_{R,t}(r_2,z_2)\phi_{R,t}^2(s,y)\right. \\
&\qquad \left.\times \|X_{r_1,z_1,r_2,z_2}(s,y)\|_{q/2}dz_1dz_2dydr_1dr_2ds\right)^{\frac{q}{2}},
\end{aligned}
\tag{6.31}
$$

where

$$
X_{r_1,z_1,r_2,z_2}(s,y) := \left(g'_{s,y}\right)^2 D_{r_1,z_1}u(s,y)D_{r_2,z_2}u(s,y)g_{r_1,z_1}g_{r_2,z_2}.
$$

Using Hölder's inequality, the Lipschitz property of g, the estimate (5.25) for all $p \ge 2$, we have

$$
\|X_{r_1,z_1,r_2,z_2}(s,y)\|_{q/2} \le C_t p_{s-r_1}(y-z_1)p_{s-r_2}(y-z_2).
$$

Plugging this bound in the estimate (6.31), we see that

$$
\begin{aligned}
K_3 \le \frac{C_t}{\varepsilon^q}\Big(&\int_{t_\varepsilon\alpha}^t\int_{t_\varepsilon\alpha}^s\int_{t_\varepsilon\alpha}^s\int_{\mathbb{R}^3}\phi_{R,t}(r_1,z_1)\phi_{R,t}(r_2,z_2)\phi_{R,t}^2(s,y) \\
&\times p_{s-r_1}(y-z_1)p_{s-r_2}(y-z_2)dz_1dz_2dydr_1dr_2ds\Big)^{\frac{q}{2}}.
\end{aligned}
\tag{6.32}
$$

Integrating in z_1 and z_2, and using the semigroup property, we have for $t_\alpha < s < t$ and for $R \geq R_0$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\prod_{i=1,2} \phi_{R,t}(r_i, z_i) \phi_{R,t}(s,y) p_{s-r_i}(y - z_i) \right) dz_1 dz_2 dy \\
&\quad = \frac{1}{\sigma_{R,t}^2} \int_{\mathbb{R}} \int_{Q_R^2} \prod_{i=1,2} \phi_{R,t}(s,y) p_{t+s-2r_i}(y - x_i) dx_1 dx_2 dy \\
&\quad \leq \frac{1}{\sigma_{R,t}^2} \int_{\mathbb{R}} \phi_{R,t}^2(s,y) \left(\prod_{i=1,2} \int_{\mathbb{R}} p_{t+s-2r_i}(y-x) dx \right) dy \\
&\quad = \frac{1}{\sigma_{R,t}^2} \int_{\mathbb{R}} \phi_{R,t}^2(s,y) dy \leq \frac{C_t}{\sigma_{R,t}^2} \leq C_t,
\end{aligned}$$

where we use Lemma A.9 part (a) of the Appendix. Now, plugging this estimate in (6.32), we get

$$K_3 \leq \frac{C_T}{\varepsilon^q} \left(\int_{t_{\varepsilon^\alpha}}^t \int_{t_{\varepsilon^\alpha}}^s \int_{t_{\varepsilon^\alpha}}^s dr_1 dr_2 ds \right)^{\frac{q}{2}} = C_T \varepsilon^{(\frac{3}{2}\alpha - 1)q}. \tag{6.33}$$

Now, choosing $\alpha = 4/5$, we get from (6.32), (6.30) and (6.33),

$$\sup_{R \geq R_0} \mathrm{P}\left(D_{v_{R,t}} F_{R,t} \right) < \varepsilon \big) \leq C_T \varepsilon^{q/5}.$$

Finally, using this estimate we get

$$\begin{aligned}
\sup_{R \geq R_0} \mathrm{E}\left[\left(D_{v_{R,t}} F_{R,t} \right)^{-p} \right] &= \sup_{R \geq R_0} p \int_0^\infty \varepsilon^{-p-1} \mathrm{P}\left(D_{v_{R,t}} F_{R,t} < \varepsilon \right) d\varepsilon \\
&\leq 1 + \sup_{R \geq R_0} p \int_0^1 \varepsilon^{-p-1} \mathrm{P}\left(D_{v_{R,t}} F_{R,t} < \varepsilon \right) d\varepsilon \\
&\leq 1 + C_T p \int_0^1 \varepsilon^{-p-1+q/5} d\varepsilon < \infty
\end{aligned}$$

for $q > 5p$, which completes our proof.

$\square$

Theorem 6.4. Let $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the mild solution to the stochastic heat equation (5.1). Assume that g satisfies hypothesis in Proposition 5.22. Suppose also that for some $q > 10$,

$\mathrm{E}\left[|g(u(t,0))|^{-q}\right] < \infty$. Fix $t > 0$ and let $F_{R,t}$ be defined as in (6.2). Then, for all $R > 0$,

$$\sup_{x\in\mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{R}},$$

where $f_{F_{R,t}}$ and ϕ are the densities of $F_{R,t}$ and $N(0,1)$, respectively.

Proof of Theorem 6.4. We will apply Theorem 3.13 to the random variable $F_{R,t} = \delta(v_{R,t})$. Fix $t > 0$. From Theorem 6.2, we have

$$\left\| \sqrt{\operatorname{Var}\left[D_{v_{R,t}} F_{R,t}\right]} \right\|_2 \leq \frac{C_t}{\sqrt{R}}. \tag{6.34}$$

We are only left to estimate the term $\left\| D_{v_{R,t}} \left(D_{v_{R,t}} F_{R,t} \right) \right\|_2$. Recall that

$$D_{v_{R,t}} F_{R,t} = \frac{1}{\sigma_{R,t}} \int_0^t \int_{\mathbb{R}} \int_{Q_R} \phi_{R,t}(s,y) g(u(s,y)) D_{s,y} u(t,x) dx dy ds.$$

Taking the Malliavin derivative, we get

$$\begin{aligned} D_{r,z}\left(D_{v_{R,t}} F_{R,t}\right) = &\frac{1}{\sigma_{R,t}} \int_r^t \int_{\mathbb{R}} \int_{Q_R} \phi_{R,t}(s,y) g'(u(s,y)) D_{r,z} u(s,y) D_{s,y} u(t,x) dx dy ds \\ &+ \frac{1}{\sigma_{R,t}} \int_0^t \int_{\mathbb{R}} \int_{Q_R} \phi_{R,t}(s,y) g(u(s,y)) D_{r,z} D_{s,y} u(t,x) dx dy ds, \end{aligned}$$

and, using the notation (5.31), we get

$$\begin{aligned} &D_{v_{R,t}}\left(D_{v_{R,t}} F_{R,t}\right) \\ &= \frac{1}{\sigma_{R,t}} \int_0^t \int_r^t \int_{\mathbb{R}^2} \int_{Q_R} \phi_{R,t}(r,z) \phi_{R,t}(s,y) g_{r,z} g'_{s,y} D_{r,z} u(s,y) D_{s,y} u(t,x) dx dy dz ds dr \\ &\quad + \frac{2}{\sigma_{R,t}} \int_0^t \int_r^t \int_{\mathbb{R}^2} \int_{Q_R} \phi_{R,t}(r,z) \phi_{R,t}(s,y) g_{r,z} g_{s,y} D_{r,z} D_{s,y} u(t,x) dx dy dz ds dr. \end{aligned}$$

Now using (5.24) and (5.26) for $D_{s,y}u(t,x)$ and $D_{r,z}D_{s,y}u(t,x)$, respectively, we have

$$D_{v_{R,t}}\left(D_{v_{R,t}}F_{R,t}\right) = 2\mathscr{Y}^1_{R,t} + \mathscr{Y}^2_{R,t} + 2\mathscr{Y}^3_{R,t} + 2\mathscr{Y}^4_{R,t},$$

where:

$$\mathscr{Y}^1_{R,t} = \int_0^t \int_r^t \int_{\mathbb{R}^2} dydzdsdr \phi^2_{R,t}(s,y)\phi_{R,t}(r,z) g_{r,z} g_{s,y} g'_{s,y} D_{r,z}u(s,y),$$

$$\begin{aligned}
\mathscr{Y}^2_{R,t} &= \int_0^t \int_r^t \int_{\mathbb{R}^2} dydzdsdr \phi_{R,t}(s,y)\phi_{R,t}(r,z) g_{r,z} g'_{s,y} D_{r,z}u(s,y) \\
&\qquad \times \int_{[s,t]\times\mathbb{R}} \phi_{R,t}(\tau,\xi) g'_{\tau,\xi} D_{s,y}u(\tau,\xi) W(d\tau,d\xi), \\
\mathscr{Y}^3_{R,t} &= \int_0^t \int_r^t \int_{\mathbb{R}^2} dydzdsdr \phi_{R,t}(s,y)\phi_{R,t}(r,z) g_{r,z} g_{s,y} \\
&\qquad \times \int_{[s,t]\times\mathbb{R}} \phi_{R,t}(\tau,\xi) g''_{\tau,\xi} D_{r,z}u(\tau,\xi) D_{s,y}u(\tau,\xi) W(d\tau,d\xi), \\
\mathscr{Y}^4_{R,t} &= \int_0^t \int_r^t \int_{\mathbb{R}^2} dydzdsdr \phi_{R,t}(s,y)\phi_{R,t}(r,z) g_{r,z} g_{s,y} \\
&\qquad \times \int_{[s,t]\times\mathbb{R}} \phi_{R,t}(\tau,\xi) g'_{\tau,\xi} D_{r,z} D_{s,y}u(\tau,\xi) W(d\tau,d\xi).
\end{aligned}$$

Putting together the terms $\mathscr{Y}^i_{R,t}$ for $i = 2,3,4$, we can write

$$D_{v_{R,t}}\left(D_{v_{R,t}}F_{R,t}\right) = 2\mathscr{Y}^1_{R,t} + \mathscr{Y}^5_{R,t},$$

where

$$\mathscr{Y}^5_{R,t} = \int_0^t \int_{\mathbb{R}} \left(\int_0^\tau \int_r^\tau \int_{\mathbb{R}^2} \phi_{R,t}(s,y)\phi_{R,t}(r,z) Z_{r,z,s,y}(\tau,\xi) dsdrdydz \right) \phi_{R,t}(\tau,\xi) W(d\tau,d\xi),$$

and we are using the notation

$$
\begin{aligned}
Z_{r,z,s,y}(\tau,\xi) =: \, & g_{r,z} g'_{s,y} g'_{\tau,\xi} D_{r,z} u(s,y) D_{s,y} u(\tau,\xi) \\
& + 2 g_{r,z} g_{s,y} g''_{\tau,\xi} D_{r,z} u(\tau,\xi) D_{s,y} u(\tau,\xi) \\
& + 2 g_{r,z} g_{s,y} g'_{\tau,\xi} D_{r,z} D_{s,y} u(\tau,\xi).
\end{aligned} \tag{6.35}
$$

Therefore,

$$
\| D_{v_{R,t}} \left(D_{v_{R,t}} F_{R,t} \right) \|_2 \leq 2 \| \mathscr{Y}^1_{R,t} \|_2 + \| \mathscr{Y}^5_{R,t} \|_2 .
$$

Estimation of $\left\| \mathscr{Y}^1_{R,t} \right\|_2$: Note that using the estimate (5.25) and Hölder's inequality we have, for $r < s$,

$$
\left\| g_{r,z} g_{s,y} g'_{s,y} D_{r,z} u(s,y) \right\|_2 \leq C_t p_{s-r}(z-y).
$$

As a consequence,

$$
\left\| \mathscr{Y}^1_{R,t} \right\|_2 \leq C_t \int_0^t \int_r^t \int_{\mathbb{R}^2} \phi^2_{R,t}(s,y) \phi_{R,t}(r,z) p_{s-r}(z-y) dydzdsdr.
$$

Integrating in z and using the semigroup property, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \phi_{R,t}(r,z) p_{s-r}(z-y) dz &= \frac{1}{\sigma_{R,t}} \int_{Q_R} \int_{\mathbb{R}} p_{t-r}(x-z) p_{s-r}(z-y) dzdx \\
&= \frac{1}{\sigma_{R,t}} \int_{Q_R} p_{t+s-2r}(x-y) dx \leq \frac{1}{\sigma_{R,t}} .
\end{aligned}
$$

Using the above estimate, and Lemma A.9 part (a), Lemma 6.1 and we get, for $R \geq R_0$,

$$
\left\| \mathscr{Y}^1_{R,t} \right\|_2 \leq \frac{C_t}{\sigma_{R,t}} \int_0^t \int_r^t \int_{\mathbb{R}} \phi^2_{R,t}(s,y) dydsdr \leq \frac{C_t}{\sigma_{R,t}} \leq \frac{C_t}{\sqrt{R}} .
$$

Estimation of $\left\| \mathscr{Y}^5_{R,t} \right\|_2$: Using the Itô-Walsh isometry of the stochastic integral in Proposition 4.8

and Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\left\|\mathscr{Y}^5_{R,t}\right\|_2^2 &= \int_0^t \int_{\mathbb{R}} \mathrm{E}\left[\left(\int_0^\tau \int_r^\tau \int_{\mathbb{R}^2} \phi_{R,t}(s,y)\phi_{R,t}(r,z) Z_{r,z,s,y}(\tau,\xi)\, ds dr dy dz\right)^2\right] \phi^2_{R,t}(\tau,\xi)\, d\xi d\tau \\
&= \int_0^t \int_{\mathbb{R}} \int_{\substack{0\le r_1\le s_1\le \tau \\ 0\le r_2\le s_2\le\tau}} \int_{\mathbb{R}^4} \prod_{i=1,2} dy_i dz_i dr_i ds_i \phi_{R,t}(s_i,y_i)\phi_{R,t}(r_i,z_i) \\
&\qquad \times \|Z_{r_i,z_i,s_i,y_i}(\tau,\xi)\|_2 \phi^2_{R,t}(\tau,\xi)\, d\xi d\tau.
\end{aligned}
$$

From the decomposition (6.35), using Hölder's inequality and the estimates (5.25) and (5.27), we can write

$$
\begin{aligned}
\left\|\mathscr{Y}^5_{R,t}\right\|_2^2 \le C_t \int_0^t \int_{\mathbb{R}} d\xi d\tau \phi^2_{R,t}(\tau,\xi) \int_{\substack{0\le r_1\le s_1\le \tau \\ 0\le r_2\le s_2\le\tau}} \int_{\mathbb{R}^4} \prod_{i=1,2} dy_i dz_i dr_i ds_i \phi_{R,t}(s_i,y_i)\phi_{R,t}(r_i,z_i) \\
\times \left[p_{s_i-r_i}(y_i-z_i)p_{\tau-s_i}(\xi-y_i) + p_{\tau-r_i}(\xi-z_i)p_{\tau-s_i}(\xi-y_i) + \Phi_{r_i,z_i,s_i,y_i}(\tau,\xi)\right].
\end{aligned}
$$

The estimates $\phi_{R,t}(r_i,z_i), \phi_{R,t}(r_i,z_i) \le \frac{1}{\sigma_{R,t}}$ imply

$$
\begin{aligned}
\left\|\mathscr{Y}^5_{R,t}\right\|_2^2 \le \frac{C_t}{\sigma^2_{R,t}} \int_0^t \int_{\mathbb{R}} d\xi d\tau \phi^2_{R,t}(\tau,\xi) \int_{\substack{0\le r_1\le s_1\le \tau \\ 0\le r_2\le s_2\le\tau}} \int_{\mathbb{R}^4} \prod_{i=1,2} dy_i dz_i dr_i ds_i \\
\times \left[p_{s_i-r_i}(y_i-z_i)p_{\tau-s_i}(\xi-y_i) + p_{\tau-r_i}(\xi-z_i)p_{\tau-s_i}(\xi-y_i) + \Phi_{r_i,z_i,s_i,y_i}(\tau,\xi)\right].
\end{aligned}
$$

Integrating the variables z_i and y_i for $i=1,2$ and using Lemma A.7, we have

$$
\left\|\mathscr{Y}^5_{R,t}\right\|_2^2 \le \frac{C}{\sigma^2_{R,t}} \int_0^t \int_{\mathbb{R}} \phi^2_{R,t}(\tau,\xi) \left(\int_{0<r<s<\tau} \left(1+(s-r)^{-1/4}\right) dr ds\right)^2 d\xi d\tau.
$$

Using the above estimate, Lemma A.9 part (a), and Lemma 6.1, we finally have for $R \ge R_0$

$$
\left\|\mathscr{Y}^5_{R,t}\right\|_2 \le \frac{C_t}{\sqrt{R}}. \tag{6.36}
$$

Finally, plugging the estimates (6.17), (6.34) and (6.36) into (3.6) we complete the proof. □

6.2 Dirac delta initial condition in PAM

Let $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the solution to equation (5.9) in dimension 1 with the initial condition $u_0 \equiv \delta_0$. The process u is no longer stationary in the space variable, but if we define U as

$$U(t,x) := \frac{u(t,x)}{p_t(x)}$$

for $(t,x) \in (0,\infty) \times \mathbb{R}$, then for any $t > 0$, the process $\{U(t,x) : x \in \mathbb{R}\}$ is stationary, see [1]. Moreover, $\lim_{t\downarrow 0} U(t,x) = 1$ in $L^p(\Omega)$ for all $x \in \mathbb{R}$ and $p \geq 2$ and the mild form (5.6) of the equation can be reformulated in terms of U as follows

$$U(t,x) = 1 + \int_{[0,t]\times\mathbb{R}} p_{\frac{\tau(t-\tau)}{t}}(\xi - \frac{\tau}{t}x) U(s,y) W(d\tau, d\xi). \tag{6.37}$$

Let

$$\varphi_{R,t}(s,y) := \frac{1}{\Sigma_{R,t}} \int_{Q_R} p_{s(t-s)/t}(y - \frac{s}{t}x) dx. \tag{6.38}$$

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left(\int_{-R}^{R} U(t,x) dx - 2R \right), \text{ where } \Sigma_{R,t}^2 := \operatorname{Var}\left[\int_{-R}^{R} U(t,x) dx \right] \tag{6.39}$$

According to Chen, Hu and Nualart [14, Proposition 5.1], for any $t > 0$ and any $x \in \mathbb{R}$, the random variable $u(t,x)$ belongs to the Sobolev space $\mathbb{D}^{k,p}$ for any $k \geq 1$ and $p \geq 2$. As a consequence, for all $t > 0$ and $x \in \mathbb{R}$, $U(t,x) \in \cap_{k\geq 1} \cap_{p\geq 2} \mathbb{D}^{k,p}$. Furthermore, for almost all $(s,y) \in (0,t) \times \mathbb{R}$, using (4.9) and (6.37), we have,

$$D_{s,y} U(t,x) = p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x) U(s,y) + \int_{[s,t]\times\mathbb{R}} p_{\frac{\tau(t-\tau)}{t}}(\xi - \frac{\tau}{t}x) D_{s,y} U(\tau,\xi) W(d\tau, d\xi), \tag{6.40}$$

and for almost all $r \le s \le t$ and $y, z \in \mathbb{R}$,

$$
\begin{aligned}
D_{r,z} D_{s,y} U(t,x) = & p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x) D_{r,z} U(s,y) \\
& + \int_{[s,t]\times\mathbb{R}} p_{\frac{\tau(t-\tau)}{t}}(\xi - \frac{\tau}{t}x) D_{r,z} D_{s,y} U(\tau,\xi) W(d\tau, d\xi).
\end{aligned} \tag{6.41}
$$

Let $G_{R,t}$ and $\Sigma_{R,t}$ be as defined in (6.39). Then, for any fixed $t > 0$, $G_{R,t} = \delta(w_{R,t})$, where

$$
\begin{aligned}
w_{R,t}(s,y) &= \mathbf{1}_{[0,t]}(s) \frac{1}{\Sigma_{R,t}} \int_{Q_R} p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x) U(s,y) dx \\
&= \mathbf{1}_{[0,t]}(s) \varphi_{R,t}(s,y) U(s,y),
\end{aligned} \tag{6.42}
$$

and $\varphi_{R,t}(s,y)$ has been defined in (6.38). Finally, we also note that

$$
D_{s,y} G_{R,t} = \frac{1}{\Sigma_{R,t}} \int_{Q_R} D_{s,y} U(t,x),
$$

and using (6.42)

$$
D_{w_{R,t}} G_{R,t} = \frac{1}{\Sigma_{R,t}} \int_0^t \int_{\mathbb{R}} \int_{Q_R} \varphi_{R,t}(s,y) U(s,y) D_{s,y} U(t,x) dx dy ds. \tag{6.43}
$$

Dividing by the factor $p_t(x)$ and using the identity (A.6) we derive the corresponding estimates for the process $U(t,x)$:

$$
\sup_{(t,x)\in[0,T]\times\mathbb{R}} \|U(t,x)\|_p \le c_{T,p}, \tag{6.44}
$$

$$
\left\| D_{s,y} U(t,x) \right\|_p \le c_{T,p} p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x). \tag{6.45}
$$

and

$$
\left\| D_{r,z} D_{s,y} u(t,x) \right\|_p \le c_{T,p} p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x) p_{\frac{r(s-r)}{s}}(z - \frac{r}{s}y). \tag{6.46}
$$

The next proposition ensures the existence of negative moments required in the application of Theorem 3.13.

Proposition 6.5. Fix $t \in (0,T]$, $p \geq 2$ and $\gamma > 5$. Then, there exist $R_0 > 1$ and a constant $c_{t,p,\gamma}$, depending on t, p and γ, such that

$$\left\| \left(D_{w_{R,t}} G_{R,t}\right)^{-1} \right\|_p \leq c_{t,p,\gamma} (\log R)^{\gamma}$$

for all $R \geq R_0$.

Proof. Using (6.43) and (6.40), we have

$$\begin{aligned} D_{w_{R,t}} G_{R,t} &= \frac{1}{\Sigma_{R,t}} \int_0^t \int_{\mathbb{R}} \int_{Q_R} \varphi_{R,t}(s,y) U(s,y) D_{s,y} U(t,x) dx dy ds \\ &= \int_0^t \int_{\mathbb{R}} \varphi_{R,t}^2(s,y) U^2(s,y) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \varphi_{R,t}(s,y) U(s,y) \left(\int_{[s,t]\times\mathbb{R}} \varphi_{R,t}(\tau,\xi) D_{s,y} U(\tau,\xi) W(d\tau,d\xi) \right) dy ds. \end{aligned}$$

Since U and DU are non-negative, $D_{w_{R,t}} G_{R,t} \geq 0$ and we have

$$\begin{aligned} D_{w_{R,t}} G_{R,t} &\geq \int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \varphi_{R,t}^2(s,y) U^2(s,y) dy ds \\ &\quad + \int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \varphi_{R,t}(s,y) U(s,y) \left(\int_{[s,t]\times\mathbb{R}} \varphi_{R,t}(\tau,\xi) D_{s,y} U(\tau,\xi) W(d\tau,d\xi) \right) dy ds \\ &=: I_1 + I_2, \end{aligned}$$

where $t_{\varepsilon^\alpha} = t - \varepsilon^{\alpha}$, with $\varepsilon \in (0, \frac{t}{2}]$ and $\alpha \in (0,1]$. As in the proof of Proposition 6.3, we can write

$$\mathrm{P}\left(D_{w_{R,t}} G_{R,t} < \varepsilon\right) \leq \mathrm{P}\left(I_1 < 2\varepsilon\right) + \mathrm{P}\left(|I_2| > \varepsilon\right). \tag{6.47}$$

We now estimate these probabilities in two steps.

Step 1: By Chebyshev inequality, for any $q \geq 2$,

$$\mathrm{P}\left(I_1 < 2\varepsilon\right) \leq \mathrm{P}\left(I_1^{-1} > \frac{1}{2\varepsilon}\right) \leq (2\varepsilon)^q \mathrm{E}\left[\left(\int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \varphi_{t,R}^2(s,y) U^2(s,y) dy ds \right)^{-q} \right]. \tag{6.48}$$

Set

$$m(\varepsilon,R)=\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\varphi_{t,R}^2(s,y)dyds.$$

Using Lemma A.9 part (b), taking into account that $s>\frac{t}{2}$, for all $R\geq R_0$, we have

$$m(\varepsilon,R)\geq\frac{c_t\varepsilon^\alpha}{\log R}. \tag{6.49}$$

Then, because the function $x\to x^{-q}$ is convex, applying Jensen's inequality, we can write

$$\begin{aligned}&\mathrm{E}\left[\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\varphi_{t,R}^2(s,y)U^2(s,y)dyds\right)^{-q}\right]\\&\leq m(\varepsilon,R)^{-q-1}\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\varphi_{R,t}^2(s,y)\mathrm{E}\left[U^{-2q}(s,y)\right]dyds.\end{aligned} \tag{6.50}$$

Since $\{U(s,y):y\in\mathbb{R}\}$ is stationary, we have for all $s\in[\frac{t}{2},2]$

$$\begin{aligned}\mathrm{E}\left[(U(s,y))^{-2q}\right]&=\mathrm{E}\left[(U(s,0))^{-2q}\right]=(p_s(0))^{2q}\mathrm{E}\left[(u(s,0))^{-2q}\right]\\&\leq(\pi t)^{-q}\mathrm{E}\left[\left(\inf_{s\in[\frac{t}{2},t]}u(s,0)\right)^{-2q}\right]=c_{t,q}<\infty,\end{aligned} \tag{6.51}$$

where $c_{t,q}$ is a constant depending on q and t and the last equality follows from [13, Theorem 1.4]. In what follows, $c_{t,q}$ will denote a generic constant depending on q and t. Substituting (6.51) into (6.50) and using Lemma A.9 part (b) and (6.49) yields

$$\begin{aligned}\mathrm{E}\left[\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\varphi_{t,R}^2(s,y)U^2(s,y)dyds\right)^{-q}\right]&\leq c_{t,q}m(\varepsilon,R)^{-q-1}\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\varphi_{R,t}^2(s,y)dyds\\&\leq c_{t,q}m(\varepsilon,R)^{-q-1}\int_{t_\varepsilon\alpha}^{t}\frac{1}{s\log R}ds\\&\leq c_{t,q}\varepsilon^{-\alpha q}(\log R)^q,\end{aligned} \tag{6.52}$$

for $R \geq R_0$. Finally, from (6.48) and (6.52), we get

$$\mathrm{P}\left(I_1 < 2\varepsilon\right) \leq c_{t,q} \left(\log R\right)^q \varepsilon^{q(1-\alpha)}. \tag{6.53}$$

Step 2: Set $\Pi = \mathrm{P}\left(|I_2| > \varepsilon\right)$. Using Fubini's theorem and Chebyschev's inequality for any $q \geq 2$, we have

$$\Pi \leq \frac{1}{\varepsilon^q}\mathrm{E}\left[\left|\int_{[t_\varepsilon\alpha, t]\times\mathbb{R}} \left(\int_{\mathbb{R}}\int_{t_\varepsilon\alpha}^{\tau} \varphi_{R,t}(\tau,\xi)\varphi_{R,t}(s,y)U(s,y)D_{s,y}U(\tau,\xi)dsdy\right)W(d\tau,d\xi)\right|^q\right].$$

Then, applying Burkholder-Davis-Gundy inequality, followed by Minkowski's inequality, we get for any $q \geq 2$

$$\begin{aligned}
\Pi &\leq \frac{c_q}{\varepsilon^q}\mathrm{E}\left[\left(\int_{t_\varepsilon\alpha}^{t}\int_{\mathbb{R}}\left(\int_{t_\varepsilon\alpha}^{\tau}\int_{\mathbb{R}}\varphi_{R,t}(\tau,\xi)\varphi_{R,t}(s,y)U(s,y)D_{s,y}U(\tau,\xi)dyds\right)^2 d\xi d\tau\right)^{\frac{q}{2}}\right] \\
&= \frac{c_q}{\varepsilon^q}\mathrm{E}\Bigg[\Bigg(\int_{t_\varepsilon\alpha}^{t}\int_{t_\varepsilon\alpha}^{\tau}\int_{t_\varepsilon\alpha}^{\tau}\int_{\mathbb{R}^3}\varphi_{R,t}^2(\tau,\xi)\varphi_{R,t}(s_1,y_1)\varphi_{R,t}(s_2,y_2) \\
&\qquad\qquad \times Y_{s_1,y_1,s_2,y_2}(\tau,\xi)dy_1dy_2d\xi ds_1ds_2d\tau\Bigg)^{\frac{q}{2}}\Bigg] \\
&\leq \frac{c_q}{\varepsilon^q}\Bigg(\int_{t_\varepsilon\alpha}^{t}\int_{t_\varepsilon\alpha}^{\tau}\int_{t_\varepsilon\alpha}^{\tau}\int_{\mathbb{R}^3}\varphi_{R,t}^2(\tau,\xi)\varphi_{R,t}(s_1,y_1)\varphi_{R,t}(s_2,y_2) \\
&\qquad\qquad \times \|Y_{s_1,y_1,s_2,y_2}(\tau,\xi)\|_{q/2}dy_1dy_2d\xi ds_1ds_2d\tau\Bigg)^{\frac{q}{2}},
\end{aligned} \tag{6.54}$$

where

$$Y_{s_1,y_1,s_2,y_2}(\tau,\xi) := U(s_1,y_1)D_{s_1,y_1}U(\tau,\xi)U(s_2,y_2)D_{s_2,y_2}U(\tau,\xi).$$

Note that using the estimates (6.44) and (6.45) and Hölder's inequality, we can write

$$\left\|Y_{s_1,y_1,s_2,y_2}(\tau,\xi)\right\|_{q/2} \leq c_{t,q}\, p_{\frac{s_1(\tau-s_1)}{\tau}}(y_1 - \frac{s_1}{\tau}\xi)\, p_{\frac{s_2(\tau-s_2)}{\tau}}(y_2 - \frac{s_2}{\tau}\xi). \tag{6.55}$$

Substituting the estimate (6.55) into (6.54), we obtain

$$\Pi \leq \frac{c_{t,q}}{\varepsilon^q}\left(\int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \varphi_{R,t}^2(\tau,\xi)\left(\int_{t_{\varepsilon^\alpha}}^\tau \int_{\mathbb{R}} \varphi_{R,t}(s,y) p_{\frac{s(\tau-s)}{\tau}}(y-\frac{s}{\tau}\xi) dy ds\right)^2 d\xi d\tau\right)^{q/2}. \tag{6.56}$$

Using the semigroup property, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi_{R,t}(s,y) p_{\frac{s(\tau-s)}{\tau}}(y-\frac{s}{\tau}\xi) dy \leq \frac{1}{\Sigma_{R,t}} \int_{\mathbb{R}^2} p_{\frac{s(t-s)}{t}}(y-\frac{s}{t}x) p_{\frac{s(\tau-s)}{\tau}}(y-\frac{s}{\tau}\xi) dy dx \\
&= \frac{1}{\Sigma_{R,t}} \int_{\mathbb{R}} p_{\frac{s(t-s)}{t}+\frac{s(\tau-s)}{\tau}}(\frac{s}{t}x-\frac{s}{\tau}\xi) dx = \frac{t}{s\Sigma_{R,t}} \int_{\mathbb{R}} p_{\frac{t(t-s)}{s}+\frac{t^2(\tau-s)}{s\tau}}(x-\frac{t}{\tau}\xi) dx = \frac{t}{s\Sigma_{R,t}},
\end{aligned}$$

where we used the identity $p_t(ax) = \frac{1}{a} p_{t/a^2}(x)$. Hence, taking into account that $t_\alpha > \frac{t}{2}$, we can write

$$\int_{t_{\varepsilon^\alpha}}^\tau \int_{\mathbb{R}} \varphi_{R,t}(s,y) p_{\frac{s(\tau-s)}{\tau}}(y-\frac{s}{\tau}\xi) dy ds \leq \frac{1}{\Sigma_{R,t}} \int_{t_{\varepsilon^\alpha}}^\tau \frac{t}{s} ds \leq \frac{2\varepsilon^\alpha}{\Sigma_{R,t}}. \tag{6.57}$$

Finally, plugging the estimate (6.57) into (6.56), and using Lemma 6.6 and Lemma A.9 part (b), we get for $R \geq R_0$,

$$\begin{aligned}
\Pi &\leq c_{t,q} \varepsilon^{q(\alpha-1)} (R\log R)^{-q/2} \left(\int_{t_{\varepsilon^\alpha}}^t \int_{\mathbb{R}} \varphi_{R,t}^2(\tau,\xi) d\xi d\tau\right)^{q/2} \\
&\leq c_{t,q} R^{-q/2} \varepsilon^{(\frac{3\alpha}{2}-1)q}.
\end{aligned} \tag{6.58}$$

Now, choosing $\alpha = 4/5$, we get, substituting (6.58) and (6.53) into (6.47),

$$\mathrm{P}\left(D_{w_{R,t}} G_{R,t} < \varepsilon\right) \leq c_{t,q} (\log R)^q \varepsilon^{q/5}.$$

Using this estimate, we get

$$
\begin{aligned}
\mathrm{E}\left[\left(D_{w_{R,t}}G_{R,t}\right)^{-p}\right] &= p\int_0^\infty \varepsilon^{-p-1}\mathrm{P}\left(D_{w_{R,t}}G_{R,t}<\varepsilon\right)d\varepsilon \\
&\le 1+p\int_0^1 \varepsilon^{-p-1}\mathrm{P}\left(D_{w_{R,t}}G_{R,t}<\varepsilon\right)d\varepsilon \\
&\le 1+c_{t,q}(\log R)^q p\int_0^1 \varepsilon^{-p-1+q/5}d\varepsilon.
\end{aligned}
$$

Finally, for $q=\gamma p>5p$, and for $R\ge R_0$, we obtain

$$
\left\|\left(D_{w_{R,t}}G_{R,t}\ \right)^{-1}\right\|_p \le c_{t,p,\gamma}(\log R)^\gamma,
$$

which completes our proof.

$\square$

Lemma 6.6. Let $\Sigma^2_{R,t}$ be as defined in (6.39). Then

$$
\lim_{R\to\infty}\frac{\Sigma^2_{R,t}}{R\log R}=2t.
$$

Theorem 6.7. For every $t>0$, there exist $C_t=C(t)>0$ and $R_0=R_0(t)>e$ such that for all $R\ge R_0$,

$$
\mathrm{d_{TV}}\left(G_{R,t},N\right)\le C_t\sqrt{\frac{\log R}{R}} \tag{6.59}
$$

Theorem 6.8. Assume that the random field $u=\{u(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}\}$ solves the parabolic Anderson model (5.9) in dimension 1 with the initial condition $u_0(x)=\delta_0$. Let $G_{R,t}$ be defined as in (6.39). Fix $\gamma>\frac{19}{2}$. Then, there exists an $R_0\ge 1$ such that for all $R\ge R_0$

$$
\sup_{x\in\mathbb{R}}|f_{G_{R,t}}(x)-\phi(x)|\le\frac{C_t(\log R)^\gamma}{\sqrt{R}},
$$

where $f_{G_{R,t}}$ and ϕ are the densities of $G_{R,t}$ and $N(0,1)$, respectively.

Proof of Theorem 6.8. We will apply Theorem 3.13 to the random variable $G_{R,t} = \delta(w_{R,t})$. Proposition 6.5 provides the estimate

$$\left\| \left(D_{w_{R,t}} G_{R,t}\right)^{-1} \right\|_4 \leq c_{t,4,\gamma} (\log R)^{\gamma}, \tag{6.60}$$

for any $\gamma > 5$, and for R large enough. Moreover, from the proof of Theorem 6.7, we have

$$\left\| \sqrt{\operatorname{Var}\left[D_{w_{R,t}} G_{R,t}\right]} \right\|_2 \leq \frac{C_t \sqrt{\log R}}{\sqrt{R}}. \tag{6.61}$$

We are only left to estimate the term $\left\| D_{w_{R,t}} \left(D_{w_{R,t}} G_{R,t}\right) \right\|_2$. Recall that from (6.43) we have

$$D_{w_{R,t}} G_{R,t} = \frac{1}{\Sigma_{R,t}} \int_0^t \int_{\mathbb{R}} \int_{Q_R} \varphi_{R,t}(s,y) U(s,y) D_{s,y} U(t,x) dx dy ds.$$

Applying again the derivative operator, we obtain

$$\begin{aligned} D_{r,z}\left(D_{w_{R,t}} G_{R,t}\right) = \frac{1}{\Sigma_{R,t}} \int_0^t \int_{\mathbb{R}} \int_{Q_R} \varphi_{R,t}(s,y) \Big(& D_{r,z} U(s,y) D_{s,y} U(t,x) \\ & + U(s,y) D_{s,y} D_{r,z} U(t,x) \Big) dx dy ds, \end{aligned}$$

so that,

$$\begin{aligned} D_{w_{R,t}}\left(D_{w_{R,t}} G_{R,t}\right) = \frac{1}{\Sigma_{R,t}} \int_{0<r<s<t} \int_{\mathbb{R}^2} \int_{Q_R} & dx dy dz ds dr \varphi_{R,t}(s,y) \varphi_{R,t}(r,z) U(r,z) \\ & \times \left(D_{r,z} U(s,y) D_{s,y} U(t,x) + 2U(s,y) D_{r,z} D_{s,y} U(t,x)\right). \end{aligned}$$

Now using (6.40) and (6.41) for $D_{s,y}U(t,x)$ and $D_{r,z}D_{s,y}U(t,x)$, we get

$$D_{w_{R,t}}\left(D_{w_{R,t}} G_{R,t}\right) = 2\mathscr{X}^1_{R,t} + \mathscr{X}^2_{R,t} + 2\mathscr{X}^3_{R,t},$$

where:

$$\mathscr{X}^1_{R,t} = \int_0^t \int_r^t \int_{\mathbb{R}^2} dzdydsdr \varphi^2_{R,t}(s,y)\varphi_{R,t}(r,z)U(r,z)U(s,y)D_{r,z}U(s,y),$$

$$\begin{aligned}\mathscr{X}^2_{R,t} &= \int_0^t \int_r^t \int_{\mathbb{R}^2} dzdydsdr \varphi_{R,t}(s,y)\varphi_{R,t}(r,z)U(r,z)D_{r,z}U(s,y) \\ &\qquad \times \int_{(s,t)\times\mathbb{R}} \varphi_{R,t}(\tau,\xi)D_{s,y}U(\tau,\xi)W(d\tau,d\xi), \\ \mathscr{X}^3_{R,t} &= \int_0^t \int_r^t \int_{\mathbb{R}^2} dzdydsdr \varphi_{R,t}(s,y)\varphi_{R,t}(r,z)U(r,z)U(s,y) \\ &\qquad \times \int_{(s,t)\times\mathbb{R}} \varphi_{R,t}(\tau,\xi)D_{r,z}D_{s,y}U(\tau,\xi)W(d\tau,d\xi).\end{aligned}$$

As a consequence, we have

$$\left\| D_{w_{R,t}}\left(D_{w_{R,t}}G_{R,t}\right)\right\|_2 \le 2\left\|\mathscr{X}^1_{R,t}\right\|_2 + \left\|\mathscr{X}^2_{R,t} + 2\mathscr{X}^3_{R,t}\right\|_2. \tag{6.62}$$

We will further estimate the two terms in the right-hand side of the previous display.

Estimation of $\left\|\mathscr{X}^1_{R,t}\right\|_2$: Using the estimates (6.44) and (6.45) and applying Hölder's inequality, we can write

$$\|U(s,y)U(r,z)D_{r,z}U(s,y)\|_2 \le C_t p_{\frac{r(s-r)}{s}}(z-\frac{r}{s}y).$$

Therefore,

$$\begin{aligned}\left\|\mathscr{X}^1_{R,t}\right\|_2 &\le \int_0^t \int_r^t \int_{\mathbb{R}^2} dzdydsdr \varphi^2_{R,t}(s,y)\varphi_{R,t}(r,z)\left\|U(s,y)U(r,z)D_{r,z}U(s,y)\right\|_2 \\ &\le C_t \int_0^t \int_r^t \int_{\mathbb{R}^2} \varphi^2_{R,t}(s,y)\varphi_{R,t}(r,z)p_{\frac{r(s-r)}{s}}(z-\frac{r}{s}y)dzdydsdr =: I_1.\end{aligned} \tag{6.63}$$

To estimate I_1, we first integrate in z and use the semigroup property, to obtain

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_{R,t}(r,z) p_{\frac{r(s-r)}{s}}(z-\frac{r}{s}y)dz &= \frac{1}{\Sigma_{R,t}} \int_{Q_R} \int_{\mathbb{R}} p_{\frac{r(t-r)}{t}}(z-\frac{r}{t}x) p_{\frac{r(s-r)}{s}}(z-\frac{r}{s}y)dzdx \\
&= \frac{1}{\Sigma_{R,t}} \int_{Q_R} p_{\frac{r(t-r)}{t}+\frac{r(s-r)}{s}}(\frac{r}{s}y-\frac{r}{t}x)dx \\
&= \frac{s}{r\Sigma_{R,t}} \int_{Q_R} p_{\frac{s^2(t-r)}{tr}+\frac{s(s-r)}{r}}(y-\frac{s}{t}x)dx. \qquad (6.64)
\end{aligned}$$

Now using the estimate $\varphi_{R,t}(s,y) \leq \frac{t}{\Sigma_{R,t}s}$ for one of the factors together with (6.64) and then applying the semigroup property in y, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi_{R,t}^2(s,y) \varphi_{R,t}(r,z) p_{\frac{r(s-r)}{s}}(z-\frac{r}{s}y)dzdy \\
&\leq \frac{t}{r\Sigma_R^3} \int_{Q_R^2} \int_{\mathbb{R}} p_{\frac{s(t-s)}{t}}(y-\frac{s}{t}x_1) p_{\frac{s^2(t-r)}{tr}+\frac{s(s-r)}{r}}(y-\frac{s}{t}x_2)dydx_1dx_2 \\
&= \frac{t}{r\Sigma_R^3} \int_{Q_R^2} p_{\frac{s(t-s)}{t}+\frac{s^2(t-r)}{tr}+\frac{s(s-r)}{r}}(\frac{s}{t}(x_1-x_2))dx_1dx_2 \\
&= \frac{t^2}{sr\Sigma_R^3} \int_{Q_R^2} p_{\frac{t(t-s)}{s}+\frac{t(t-r)}{r}+\frac{t^2(s-r)}{sr}}(x_1-x_2)dx_1dx_2 \\
&= \frac{t^2}{sr\Sigma_R^3} \int_{Q_R^2} p_{\frac{2t(t-r)}{r}}(x_1-x_2)dx_1dx_2 \\
&= \frac{4Rt^2}{\pi sr\Sigma_R^3} \int_{\mathbb{R}} \varphi(\xi) e^{-\frac{2t(t-r)}{rR^2}\xi^2} d\xi, \qquad (6.65)
\end{aligned}$$

where the last equality follows from Lemma A.10. So, substituting (6.65) into (6.63), we get

$$I_1 \leq C_t \frac{R}{\Sigma_{R,t}^3} \int_{\mathbb{R}} \varphi(\xi) \int_0^t \frac{1}{s} \int_0^s \frac{1}{r} e^{-\frac{2s(s-r)}{r}\frac{\xi^2}{R^2}} drdsd\xi.$$

By Lemma A.11, we can write

$$I_1 \leq C_t \frac{R\log R}{\Sigma_R^3} \left(\int_{\mathbb{R}} \varphi(\xi) \log(e+\frac{1}{\sqrt{2}|\xi|})d\xi \right) \left(\int_0^t \log(e+\frac{1}{s})ds \right).$$

Finally Lemma 6.6 yields

$$I_1 \leq C_t(R\log R)^{-1/2}. \qquad (6.66)$$

Estimation of $\left\|\mathscr{X}_{R,t}^2+2\mathscr{X}_{R,t}^3\right\|_2$: Define

$$V_{r,z,s,y}(\tau,\xi)=U(r,z)D_{r,z}U(s,y)D_{s,y}u(\tau,\xi)+2U(r,z)U(r,z)D_{r,z}D_{s,y}U(\tau,\xi).$$

With this notation in mind, we can write

$$\mathscr{X}_{R,t}^2+2\mathscr{X}_{R,t}^3=\int_{[0,t]\times\mathbb{R}}\left(\int_0^\tau\int_r^\tau\int_{\mathbb{R}^2}\varphi_{R,t}(s,y)\varphi_{R,t}(r,z)V_{r,z,s,y}(\tau,\xi)dsdrdydz\right)\varphi_{R,t}(\tau,\xi)W(d\tau,d\xi).$$

Using the Itô-Walsh isometry of the stochastic integral and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
I_2 =&: \|\mathscr{X}_{R,t}^2+2\mathscr{X}_{R,t}^3\|_2^2\\
&=\int_0^t\int_{\mathbb{R}}\mathrm{E}\left[\left(\int_0^\tau\int_r^\tau\int_{\mathbb{R}^2}\varphi_{R,t}(s,y)\varphi_{R,t}(r,z)V_{r,z,s,y}(\tau,\xi)dsdrdydz\right)^2\right]\\
&\quad\times\varphi_{R,t}^2(\tau,\xi)d\xi d\tau\\
&=\int_0^t\int_{\mathbb{R}}\int_{\substack{0\le r_1\le s_1\le\tau\\0\le r_2\le s_2\le\tau}}\int_{\mathbb{R}^4}\prod_{i=1,2}dy_idz_idr_ids_i\varphi_{R,t}(s_i,y_i)\varphi_{R,t}(r_i,z_i)\\
&\quad\times\|V_{r_i,z_i,s_i,y_i}(\tau,\xi)\|_2\varphi_{R,t}^2(\tau,\xi)d\xi d\tau.
\end{aligned}$$

Using (6.44) (6.45) and (6.46), we see that, for $i=1,2$,

$$\|V_{r_i,z_i,s_i,y_i}(\tau,\xi)\|_2\le C_tp_{\frac{s_i(\tau-s_i)}{\tau}}(y_i-\frac{s_i}{\tau}\xi)p_{\frac{r_i(s_i-r_i)}{s_i}}(z_i-\frac{r_i}{s_i}y_i)$$

and hence

$$\begin{aligned}
I_2\le C_t&\int_0^t\int_0^t\int_0^{s_1}\int_0^{s_2}\int_{s_1\vee s_2}^t\int_{\mathbb{R}}d\xi d\tau dr_1dr_2ds_1ds_2\varphi_{R,t}^2(\tau,\xi)\\
&\times\prod_{i=1,2}\int_{\mathbb{R}^2}\varphi_{R,t}(s_i,y_i)\varphi_{R,t}(r_i,z_i)p_{\frac{s_i(\tau-s_i)}{\tau}}(y_i-\frac{s_i}{\tau}\xi)p_{\frac{r_i(s_i-r_i)}{s_i}}(z_i-\frac{r_i}{s_i}y_i)dz_idy_i.
\end{aligned}\tag{6.67}$$

Integrating in the variable z_i and using the semigroup property, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi_{R,t}(r_i,z_i)p_{\frac{r_i(s_i-r_i)}{s_i}}(z_i-\frac{r_i}{s_i}y_i)dz_i \\
&= \frac{1}{\Sigma_{R,t}}\int_{Q_R}\int_{\mathbb{R}} p_{\frac{r_i(t-r_i)}{t}}(z_i-\frac{r_i}{t}x_i)p_{\frac{r_i(s_i-r_i)}{s_i}}(z_i-\frac{r_i}{s_i}y_i)dz_idx_i \\
&= \frac{1}{\Sigma_{R,t}}\int_{Q_R} p_{\frac{r_i(t-r_i)}{t}+\frac{r_i(s_i-r_i)}{s_i}}(\frac{r_i}{t}x_i-\frac{r_i}{s_i}y_i)dx_i \\
&= \frac{s_i}{r_i\Sigma_{R,t}}\int_{Q_R} p_{\frac{s_i^2(t-r_i)}{r_it}+\frac{s_i(s_i-r_i)}{r_i}}(\frac{s_i}{t}x_i-y_i)dx_i.
\end{aligned} \tag{6.68}$$

From (6.68), using the estimate $\varphi_{R,t}(s_i,y_i) \le \frac{t}{s_i\Sigma_{R,t}}$, and applying the semigroup property, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi_{R,t}(s_i,y_i)\varphi_{R,t}(r_i,z_i)p_{\frac{s_i(\tau-s_i)}{\tau}}(y_i-\frac{s_i}{\tau}\xi)p_{\frac{r_i(s_i-r_i)}{s_i}}(z_i-\frac{r_i}{s_i}y_i)dz_idy_i \\
&\le \frac{t}{\Sigma_{R,t}^2 r_i}\int_{Q_R}\int_{\mathbb{R}} p_{\frac{s_i^2(t-r_i)}{r_it}+\frac{s_i(s_i-r_i)}{r_i}}(\frac{s_i}{t}x_i-y_i)p_{\frac{s_i(\tau-s_i)}{\tau}}(y_i-\frac{s_i}{\tau}\xi)dy_idx_i \\
&= \frac{t}{\Sigma_{R,t}^2 r_i}\int_{Q_R} p_{\frac{s_i^2(t-r_i)}{r_it}+\frac{s_i(s_i-r_i)}{r_i}+\frac{s_i(\tau-s_i)}{\tau}}(\frac{s_i}{t}x_i-\frac{s_i}{\tau}\xi)dx_i \\
&= \frac{t\tau}{\Sigma_{R,t}^2 r_is_i}\int_{Q_R} p_{\frac{\tau^2(t-r_i)}{r_it}+\frac{\tau^2(s_i-r_i)}{r_is_i}+\frac{\tau(\tau-s_i)}{s_i}}(\frac{\tau}{t}x_i-\xi)dx_i.
\end{aligned} \tag{6.69}$$

Substituting the estimate (6.69) into (6.67), together with bound $\varphi_{R,t}(\tau,\xi) \le \frac{t}{\tau\Sigma_{R,t}}$, and then integrating in ξ this time, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi_{R,t}^2(\tau,\xi) \prod_{i=1,2} \int_{\mathbb{R}^2} \varphi_{R,t}(s_i,y_i)\varphi_{R,t}(r_i,z_i) p_{\frac{s_i(\tau-s_i)}{\tau}}(y_i - \frac{s_i}{\tau}\xi) p_{\frac{r_i(s_i-r_i)}{s_i}}(z_i - \frac{r_i}{s_i}y_i) dz_i dy_i d\xi \\
&\le \frac{t^4}{\Sigma_{R,t}^6 r_1 r_2 s_1 s_2} \int_{Q_R^2} \int_{\mathbb{R}} \prod_{i=1,2} p_{\frac{\tau^2(t-r_i)}{r_i t} + \frac{\tau^2(s_i-r_i)}{r_i s_i} + \frac{\tau(\tau-s_i)}{s_i}} (\frac{\tau}{t} x_i - \xi) dx_i d\xi \\
&= \frac{t^4}{\Sigma_{R,t}^6 r_1 r_2 s_1 s_2} \int_{Q_R^2} p_{\frac{\tau^2(t-r_1)}{r_1 t} + \frac{\tau^2(s_1-r_1)}{r_1 s_1} + \frac{\tau(\tau-s_1)}{s_1} + \frac{\tau^2(t-r_2)}{r_2 t} + \frac{\tau^2(s_2-r_2)}{r_2 s_2} + \frac{\tau(\tau-s_2)}{s_2}} (\frac{\tau}{t}(x_1 - x_2)) dx_1 dx_2 \\
&= \frac{t^5}{\Sigma_{R,t}^6 \tau r_1 r_2 s_1 s_2} \int_{Q_R^2} p_{\frac{t(t-r_1)}{r_1} + \frac{t^2(s_1-r_1)}{r_1 s_1} + \frac{t^2(\tau-s_1)}{\tau s_1} + \frac{t(t-r_2)}{r_2} + \frac{t^2(s_2-r_2)}{r_2 s_2} + \frac{t^2(\tau-s_2)}{\tau s_2}} (x_1 - x_2) dx_1 dx_2 \\
&= \frac{t^5}{\Sigma_{R,t}^6 \tau r_1 r_2 s_1 s_2} \int_{Q_R^2} p_{2t(\frac{t}{r_1} + \frac{t}{r_2} - \frac{t}{\tau} - 1)} (x_1 - x_2) dx_1 dx_2 \\
&= \frac{4 t^5 R}{\pi \Sigma_{R,t}^6 \tau r_1 r_2 s_1 s_2} \int_{\mathbb{R}} \varphi(\xi) e^{-2t(\frac{t}{r_1} + \frac{t}{r_2} - \frac{t}{\tau} - 1)\frac{\xi^2}{R^2}} d\xi,
\end{aligned} \tag{6.70}$$

where in the last inequality we have used Lemma (A.10). Moreover, using the bound

$$\frac{t}{r_1} + \frac{t}{r_2} - \frac{t}{\tau} - 1 \ge \frac{t - r_1}{2r_1} + \frac{t - r_2}{2r_2},$$

and substituting (6.70) into (6.67), we obtain

$$\begin{aligned}
I_2 &\le \frac{C_t t^5 R}{\Sigma_{R,t}^6} \int_{\mathbb{R}} \int_0^t \int_0^t \int_{r_1}^t \int_{r_2}^t \int_{s_1 \vee s_2}^{\tau} \frac{\varphi(\xi)}{\tau r_1 r_2 s_1 s_2} e^{-t(\frac{t-r_1}{r_1} + \frac{t-r_2}{r_2})\frac{\xi^2}{R^2}} d\tau ds_1 ds_2 dr_1 dr_2 d\xi \\
&\le \frac{C_t t^5 R}{\Sigma_{R,t}^6} \int_{\mathbb{R}} \varphi(\xi) d\xi \int_0^t \frac{d\tau}{\tau} \left(\int_0^{\tau} \frac{1}{s} \int_0^s \frac{1}{r} e^{-s(\frac{s-r}{r})\frac{\xi^2}{R^2}} dr ds \right)^2.
\end{aligned}$$

By Lemma A.11, we get

$$I_2 \le \frac{C_t t^5 R (\log R)^2}{\Sigma_{R,t}^6} \int_{\mathbb{R}} \varphi(\xi) \int_0^t \frac{d\tau}{\tau} \left(\int_0^{\tau} \log(e + \frac{1}{s}) \log(e + \frac{1}{|\xi|}) ds, \right)^2.$$

which implies, in view of Lemma 6.6 part (b),

$$I_2 \le C_t \frac{1}{R^2 \log R} \tag{6.71}$$

for all $R \geq R_0$. Plugging (6.66) and (6.71) into (6.62), yields, for all $R \geq R_0$,

$$\|D_{w_{R,t}}\left(D_{w_{R,t}} G_{R,t}\right)\|_2 \leq C_t (R \log R)^{-1/2}. \tag{6.72}$$

Finally, from (6.60), (6.61) and (6.72), applying Theorem 3.13 we get

$$\sup_{x \in \mathbb{R}} |f_{G_R(t)}(x) - \phi(x)| \leq \frac{C_{t,\gamma} (\log R)^{2\gamma - \frac{1}{2}}}{\sqrt{R}},$$

for all $R \geq R_0$, which yields the desired estimate. □

Chapter 7

Rate of Convergence in Breuer-Major Theorem

Breuer and Major established a normal approximation result in [8] which can be seen as a generalization of central limit theorem and usually referred to as Breuer-Major theorem, see Theorem 7.3. In this chapter we investigate the rate of convergence in total variation and Wasserstein distances associated to this normal approximation. We first introduce Breuer-Major theorem in section 7.1. We refer to the book Nourdin and Peccati [38] for the general treatment of the subject. Then we recall some estimates on the rate of convergence in a fixed Wiener chaos in section 7.2, based on Biermé, Bonami, Nourdin, and Peccati [6], Nourdin and Peccati [39]. In section 7.3 and section 7.4, we present the results in Kuzgun and Nualart [30] with their proofs. In the last part of the section 7.3, we include more results in the literature on the same problem, see Nualart and Zhou [46], Nourdin, Peccati, and Yang [40], Nourdin, Nualart, and Peccati [41]. Finally, section 7.5 is devoted to recall some technical results which are used in the proofs of the estimates given in this chapter.

7.1 Breuer-Major theorem

Definition 7.1. $X = \{X_n\}_{n\in\mathbb{N}}$ is called *centered stationary Gaussian sequence with unit variance* if X is a centered, unit variance Gaussian family of random variables defined on a probability space $(\Omega, \mathfrak{F}, \mathrm{P})$ where the covariance function $\mathrm{E}[X_k X_l]$ of X depends on k, l only through a function of $|k-l|$.

Throughout, let $\rho(k) := \mathrm{E}[X_0 X_k]$ for $k \in \mathbb{N}$ and set $\rho(k) := \rho(-k)$ for $k \in \mathbb{Z}_{<0}$.

Proposition 7.2. There exists a real separable Hilbert space $\mathfrak{H}$, and an isonormal Gaussian process over $\mathfrak{H}$, written $(X(h))_{h\in\mathfrak{H}}$, with the property that there exists a set $E = \{e_k\}_{k\in\mathbb{Z}} \subset \mathfrak{H}$ such that

(i) E is a basis for $\mathfrak{H}$,

(ii) $\langle e_k, e_l\rangle_{\mathfrak{H}} = \rho(k-l)$ for every $k,l \in \mathbb{Z}$,

(iii) $X_k = X(e_k)$ for every $k \in \mathbb{Z}$.

Let $f \in L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ has mean zero. Recall from Corollary 2.14 that f has an Hermite expansion

$$f(x) = \sum_{q=d}^{\infty} c_q H_q(x)$$

where $c_d \neq 0$ and $d \in \mathbb{N}$ is called the Hermite rank. Consider the sequence of normalized partial sums associated with the Gaussian subordinated process $\{f(X_n)\}_{n\in\mathbb{N}}$:

$$F_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(X_k), n \geq 1 \tag{7.1}$$

Further let, for $n \in \mathbb{N}$, $\sigma_n^2 = \mathrm{Var}[F_n]$ and define

$$Y_n := \frac{F_n}{\sigma_n} \tag{7.2}$$

Theorem 7.3 (Breuer-Major Theorem)**.** Let $\{F_n\}_{n\in\mathbb{N}}$ be as defined in (7.1). Assume that

$$\sum_{k\in\mathbb{Z}} |\rho(k)|^d < \infty, \tag{7.3}$$

and set

$$\sigma^2 = \sum_{q=d}^{\infty} q! c_q^2 \sum_{k\in\mathbb{Z}}^{q} \in [0, \infty). \tag{7.4}$$

Then,

$$F_n \to \sigma Z \text{ in distribution as } n \to \infty$$

where Z is a standard normal random variable.

Proof of Breuer-Major theorem first given in the seminar paper [8] by Breuer and Major. Their proof relied on the combinatorial cumulants/diagrams computations.

Now, we study convergence in total variation and wasserstein distances and in particular to obtain rate of convergence results associated to these distances in Breuer-Major theorem.

7.2 Fixed Wiener chaos

In this section we will consider the case where $f = H_d$ for some $d \geq 2$. See the following convergence in total variation result in [6].

Theorem 7.4. Let $f = H_d$ and $\rho \in l^d$ for a fixed $d \geq 2$. Then

$$\lim_{n\to\infty} \mathrm{d}_{\mathrm{TV}}(Y_n, Z) = 0.$$

Also, computing the third and fourth cumulants for the case $f = H_d$ in [6] using the optimal fourth moment result in [39] leads to the following optimal rate for fixed Wiener chaos.

Theorem 7.5. Let $f(x) = H_d(x)$. Then, for all $n \geq 1$,

$$d_{TV}(Y_n, Z) \leq \frac{C}{n}\left(\sum_{|k|\leq n} |\rho(k)|^{d-1}\right)^2 \sum_{|k|\leq n} |\rho(k)|^2 + \frac{C}{\sqrt{n}}\left(\sum_{|k|\leq n} |\rho(k)|^{3d/4}\right)^2 \mathbf{1}_{\{d \text{ even }\}} \tag{7.5}$$

with a matching lower bound. In particular, if $d = 2$, and $f(x) = H_2(x) = x^2 - 1$, then

$$\frac{c}{\sqrt{n}}\left(\sum_{|k|\leq n} |\rho(k)|^{3/2}\right)^2 d_{TV}(Y_n, Z) \leq \frac{C}{\sqrt{n}}\left(\sum_{|k|\leq n} |\rho(k)|^{3/2}\right)^2. \tag{7.6}$$

7.3 Total variation distance

The estimate (7.6) $f = H_2$ in [39], with a matching lower bound, was extended to $f \in \mathbb{D}^{6,8}(\mathbb{R},\gamma)$ in [46]. This upper bound, however, cannot be obtained as a consequence of Theorem 3.10 and requires a more intensive application of Stein's method (see [39, 46]). In this section we present the results in [30] with their proofs, and then recall recent results on the very some problem. In order to state the main theorems and give a proof, we will set up some notation and recall some preliminary results.

7.3.1 Some preliminaries

Define the shift operator T_k by

$$T_k(g)(x) = \sum_{m=d}^{\infty} c_m H_{m-k}(x)\,. \tag{7.7}$$

To simplify the notation we will write $T_k(g) = g_k$.

Suppose that F is a random variable in the first Wiener chaos of W of the form $F = I_1(\varphi)$, where $\varphi \in \mathfrak{H}$ has norm one. Then $g_k(F)$ has the representation

$$g(F) = \delta^k(g_k(F)\varphi^{\otimes k})\,. \tag{7.8}$$

Moreover, if $g(F) \in \mathbb{D}^{j,p}$ for some $j \geq 0$ and $p > 1$, then $g_k(F) \in \mathbb{D}^{j+k,p}$. We refer to [46] for the proof of these results.

Consider the isonormal Gaussian process in the proof of Corollary 2.14. That is, $\mathfrak{H} = \mathbb{R}$, the probability space $(\Omega,\mathfrak{F},\mathrm{P}) = (\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$ $W(h) = h$. For any $k \geq 0$ and $p \geq 1$, denote by $\mathbb{D}^{k,p}(\mathbb{R},\phi(x)dx)$ the corresponding Sobolev spaces of functions. Notice that if $F = I_1(\varphi)$ is an element in the first Wiener chaos with $\|\varphi\|_{\mathfrak{H}} = 1$, then $g \in \mathbb{D}^{k,p}(\phi(x)dx)$ if and only if $g(F) \in \mathbb{D}^{k,p}$.

Given a function $g \in L^2(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$ with expansion (Corollary 2.14), we denote by $A(g)$ the function in $L^2(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$, whose Hermite coefficients are the absolute values of the coefficients of g, that is,

$$A(g)(x)=\sum_{q=d}^{\infty}|c_q|H_q(x). \tag{7.9}$$

The operator A is acting on $L^2(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$ which replace the Hermite coefficients by its absolute values. Clearly, for any integer $k\geq 0$, and for any $g\in\mathbb{D}^{k,2}(\phi(x)dx)$, we have

$$\|A(g)\|_{k,2}=\|g\|_{k,2}\ .$$

Therefore, g belongs to $\mathbb{D}^{k,2}(\phi(x)dx)$ if and only if $A(g)\in\mathbb{D}^{k,2}(\phi(x)dx)$. If we consider functions in $L^p(\mathbb{R},\mathscr{B}(\mathbb{R})\phi(x)dx)$ for some real number $p>2$, we do not know whether $g\in L^p(\mathbb{R},\mathscr{B}(\mathbb{R})\phi(x)dx)$ implies $A(g)\in L^p(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$. However, the following result holds.

Lemma 7.6. Suppose that $A(g)\in\mathbb{D}^{k,2M}(\phi(x)dx)$ for some integers $M\geq 2$ and $k\geq 0$. Then $g\in\mathbb{D}^{k,2M}(\phi(x)dx)$.

Proof. We will show the result only for $k=0$, the case $k\geq 1$ being similar. Let $g=\sum_{q=d}^{\infty}c_qH_q$ and define $g_+=\sum_{q=d}^{\infty}c_q\mathbf{1}_{\{q:c_q>0\}}H_q$ and $g_-=\sum_{q=d}^{\infty}c_q\mathbf{1}_{\{q:c_q<0\}}H_q$. Then $g=g_++g_-$. We will show that $g_+\in L^{2M}(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$, and in the same way one can prove that $g_-\in L^{2M}(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$. Using Proposition 2.42, we can write

$$\begin{aligned}\mathrm{E}\left[g_+^{2M}\right]&=\lim_{N\to\infty}\mathrm{E}\left[(g_+^{(N)})^{2M}\right]\\&=\sum_{q_1,\dots,q_{2M}=0}^{\infty}\left(\prod_{i=1}^{2M}c_{q_i}\mathbf{1}_{\{q:c_q>0\}}\right)\sum_{\beta\in\mathscr{D}_q}\frac{\prod_{i=1}^{2M}q_i!}{\prod_{1\leq j<k\leq 2M}\beta_{jk}!}\,,\end{aligned}$$

where $\mathscr{D}_q$ is the set of nonnegative integers β_{jk}, $1\leq j<k\leq 2M$, satisfying $q_i=\sum_{j\,\mathrm{or}\,k=i}\beta_{jk}$, $i=1,\dots,2M$. Clearly, this implies that $\mathrm{E}\left[g_+^{2M}\right]\leq\mathrm{E}\left[A(g)^{2M}\right]<\infty$. □

The next lemma provides a criterion for a function g to satisfy $A(g)\in\mathbb{D}^{\ell,M}(\phi(x)dx)$ for integers $\ell\geq 0$, $M\geq 3$.

Lemma 7.7. Fix integers $\ell\geq 0$ and $M\geq 3$. Let g be a function in $g\in\mathbb{D}^{\ell,2}(\phi(x)dx)$, with Hermite expansion $g = \sum_{k=0}^{\infty} c_q H_q$. Then, $A(g) \in \mathbb{D}^{\ell,M}(\phi(x)dx)$ if

$$\sum_{q=0}^{\infty} |c_q| q^{\frac{\ell}{2}-\frac{1}{4}} \sqrt{q!} (M-1)^{\frac{q}{2}} < \infty. \tag{7.10}$$

Proof. We have

$$D^{\ell} A^{(N)}(g) = \sum_{q=\ell}^{N} |c_q| q(q-1)\cdots(q-\ell+1) H_{q-\ell}.$$

Applying the estimate (see, for instance, [33])

$$\|H_q\|_{L^M(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)} = c(M) q^{-\frac{1}{4}} \sqrt{q!} (M-1)^{\frac{q}{2}} (1 + O(q^{-1})),$$

we obtain

$$\begin{aligned} \|D^{\ell} A^{(N)}(g)\|_{L^M(\mathbb{R},\phi(x)dx)} &\leq c(M) \Bigg(|c_\ell| \sum_{q=\ell}^{N} |c_q| q(q-1)\cdots(q-\ell+1)(q-\ell)^{-\frac{1}{4}} \\ &\qquad \times \sqrt{(q-\ell)!} (M-1)^{\frac{q-\ell}{2}} (1+O(q^{-1})) \Bigg) \\ &\leq c(M,\ell) \Bigg(|c_\ell| + \sum_{q=\ell}^{N} |c_q| q^{\frac{\ell}{2}-\frac{1}{4}} \sqrt{q!} (M-1)^{\frac{q-\ell}{2}} (1+O(q^{-1})) \Bigg). \end{aligned}$$

Therefore, taking into account that $A^{(N)}(g)$ converges in $L^2(\Omega)$ to $A(g)$ as N tends to infinity, we conclude that $\mathrm{E}\left[(|D^{\ell} A(g)|^M)\right] < \infty$ if (7.10) holds. $\square$

7.3.2 Main result

Theorem 7.8. Assume that $f \in L^2(\mathbb{R}, \phi(x)dx)$ has Hermite rank $d \geq 2$ and satisfies $A(g) \in \mathbb{D}^{1,4}(\mathbb{R}, \phi(x)dx)$. Suppose that (7.3) holds true and let Y_n be the random variable defined in (7.2). Then we have the following estimates:

(i) If $d=2$, then

$$d_{\mathrm{TV}}(Y_n,Z) \le Cn^{-\frac{1}{2}} \left(\sum_{|k|\le n} |\rho(k)| \right)^{\frac{1}{2}} + Cn^{-\frac{1}{2}} \left(\sum_{|k|\le n} |\rho(k)|^{\frac{4}{3}} \right)^{\frac{3}{2}}. \tag{7.11}$$

(ii) If $d \ge 3$, we have

$$\begin{aligned} d_{\mathrm{TV}}(Y_n,Z) \le{}& Cn^{-\frac{1}{2}} \sum_{|k|\le n} |\rho(k)|^{d-1} \left(\sum_{|k|\le n} |\rho(k)|^2 \right)^{\frac{1}{2}} \\ &+ Cn^{-\frac{1}{2}} \left(\sum_{|k|\le n} |\rho(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{|k|\le n} |\rho(k)| \right)^{\frac{1}{2}}. \end{aligned} \tag{7.12}$$

Proof. Consider a centered stationary Gaussian family of random variables $X=\{X_n, n\ge 0\}$ with unit variance and covariance $\rho(k) = \mathrm{E}[X_0 X_k]$ for $k \ge 0$. We put $\rho(-k)=\rho(k)$ for $k<0$. Suppose that $\mathfrak{H}$ is a Hilbert space and $e_i \in \mathfrak{H}$, $i \ge 0$, are elements such that, for each $i,j\ge 0$, we have $\langle e_i, e_j\rangle_{\mathfrak{H}} = \rho(i-j)$. In this situation, if $\{W(\phi) : \phi \in \mathfrak{H}\}$ is an isonormal Gaussian process, then the sequence $X=\{X_n, n\ge 0\}$ has the same law as $\{W(e_n), n \ge 0\}$ and we can assume, without any loss of generality, that $X_n = W(e_n)$.

Consider the sequence f_n and Y_n introduced in (7.1), where $g \in L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ has Hermite rank $d \ge 2$ and let $\sigma_n^2 = \mathrm{E}\left[f_n^2\right]$. Under condition (7.3), it is well known that as $n\to\infty$, $\sigma_n^2 \to \sigma^2$, where σ^2 has been defined in (7.4). . Notice that $\sigma > 0$ implies that σ_n is bounded below for n large enough. Taking into account (7.8), we have the representation $Y_n = \delta(\frac{1}{\sigma_n} u_n)$, where

$$u_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} g_1(X_j) e_j, \tag{7.13}$$

and g_1 is the shifted function introduced in (7.7).

As a consequence of Proposition 3.10, we have the estimate

$$
\begin{aligned}
d_{TV}(Y_n,N) &\leq 2\sqrt{\operatorname{Var}(\langle DY_n, \frac{1}{\sigma_n} u_n\rangle_{\mathfrak{H}})} \\
&\leq C\sqrt{\operatorname{Var}(\langle DY_n, u_n\rangle_{\mathfrak{H}})}.
\end{aligned} \tag{7.14}
$$

Then, we can write

$$
\langle DY_n, u_n\rangle_{\mathfrak{H}} = \frac{1}{n}\sum_{i,j=1}^{n} g'(X_i)g_1(X_j)\rho(i-j).
$$

The random variable $g'(X_i)g_1(X_j)$ belongs to $L^2(\Omega,\mathfrak{F},\Omega)$, but we do not know its chaos expansion. For this reason, we need to use a limit argument. We have

$$
\langle DY_n, u_n\rangle_{\mathfrak{H}} = \lim_{N\to\infty}\Phi_{n,N},
$$

where the convergence holds in $L^1(\Omega,\mathfrak{F},\mathrm{P})$ and

$$
\Phi_{n,N} = \frac{1}{n}\sum_{i,j=1}^{n}\sum_{q_1,q_2=d}^{N} c_{q_1}c_{q_2}q_1 H_{q_1-1}(X_i)H_{q_2-1}(X_j)\rho(i-j).
$$

Therefore, by Fatou's lemma

$$
\begin{aligned}
\operatorname{Var}\left[\langle DY_n, u_n\rangle_{\mathfrak{H}}\right] &= \mathrm{E}\left[\langle DY_n, u_n\rangle_{\mathfrak{H}}^2\right] - \left(\mathrm{E}\left[\langle DY_n, u_n\rangle_{\mathfrak{H}}\right]\right)^2 \\
&\leq \liminf_{N\to\infty}\left(\mathrm{E}\left[\Phi_{n,N}^2\right] - \left(\mathrm{E}\left[\Phi_{n,N}\right)\right]^2\right) \\
&= \liminf_{N\to\infty}\operatorname{Var}\left[\Phi_{n,N}\right].
\end{aligned}
$$

We can write

$$\operatorname{Var}[\Phi_{n,N}] = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \sum_{q_1,q_2,q_3,q_4=d}^{N} q_1 q_3 c_{q_1} c_{q_2} c_{q_3} c_{q_4} \rho(i_1-i_2)\rho(i_3-i_4)$$
$$\times \operatorname{Cov}(H_{q_1-1}(X_{i_1})H_{q_2-1}(X_{i_2}), H_{q_3-1}(X_{i_3})H_{q_4-1}(X_{i_4})). \tag{7.15}$$

The next step is to compute the covariance appearing in the previous formula. To do this we will write the Hermite polynomials in terms of stochastic integrals and apply Lemma 7.17. That is,

$$\begin{aligned}
&\operatorname{Cov}\left[H_{q_1-1}(X_{i_1})H_{q_2-1}(X_{i_2}), H_{q_3-1}(X_{i_3})H_{q_4-1}(X_{i_4})\right] \\
&= \operatorname{Cov}\left[I_{q_1-1}(e_{i_1}^{\otimes(q_1-1)})I_{q_2-1}(e_{i_2}^{\otimes(q_2-1)}), I_{q_3-1}(e_{i_3}^{\otimes(q_3-1)})I_{q_4-1}(e_{i_4}^{\otimes(q_4-1)})\right] \\
&= \mathrm{E}\left[I_{q_1-1}(e_{i_1}^{\otimes(q_1-1)})I_{q_2-1}(e_{i_2}^{\otimes(q_2-1)})I_{q_3-1}(e_{i_3}^{\otimes(q_3-1)})I_{q_4-1}(e_{i_4}^{\otimes(q_4-1)})\right] \\
&\quad - \mathrm{E}\left[I_{q_1-1}(e_{i_1}^{\otimes(q_1-1)})I_{q_2-1}(e_{i_2}^{\otimes(q_2-1)})\right]\mathrm{E}\left[I_{q_3-1}(e_{i_3}^{\otimes(q_3-1)})I_{q_4-1}(e_{i_4}^{\otimes(q_4-1)})\right]
\end{aligned}$$

and using Lemma 7.17,

$$\begin{aligned}
&\mathrm{E}\left[I_{q_1-1}(e_{i_1}^{\otimes(q_1-1)})I_{q_2-1}(e_{i_2}^{\otimes(q_2-1)})I_{q_3-1}(e_{i_3}^{\otimes(q_3-1)})I_{q_4-1}(e_{i_4}^{\otimes(q_4-1)})\right] \\
&= \sum_{\beta\in\mathscr{D}_q} C_{q,\beta} \prod_{1\le j<k\le 4} \rho(i_j-i_k)^{\beta_{jk}},
\end{aligned} \tag{7.16}$$

where

$$C_{q,\beta} = \frac{\prod_{j=1}^4 (q_j-1)!}{\prod_{1\le j<k\le 4} \beta_{jk}!}$$

and $\mathscr{D}_q$ is the set of nonnegative integers β_{jk}, satisfying

$$q_\ell - 1 = \sum_{j \text{ or } k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \le \ell \le 4. \tag{7.17}$$

On the other hand,

$$
\begin{aligned}
&\mathrm{E}\left[I_{q_1-1}(e_{i_1}^{\otimes(q_1-1)})I_{q_2-1}(e_{i_2}^{\otimes(q_2-1)})\right]\mathrm{E}\left[I_{q_3-1}(e_{i_3}^{\otimes(q_3-1)})I_{q_4-1}(e_{i_4}^{\otimes(q_4-1)})\right] \\
&=(q_1-1)!(q_3-1)!\rho^{q_1-1}(i_1-i_2)\rho^{q_3-1}(i_3-i_4), \qquad (7.18)
\end{aligned}
$$

if $q_1=q_2$ and $q_3=q_4$, and zero otherwise. Notice that (7.18) is precisely the term in the sum (7.16) with $\beta_{12}=q_1-1$, $\beta_{34}=q_3-1$ and $\beta_{13}=\beta_{14}=\beta_{23}=\beta_{24}=0$. As a consequence, we obtain

$$
\mathrm{Cov}\left[H_{q_1-1}(X_{i_1})H_{q_2-1}(X_{i_2}),H_{q_3-1}(X_{i_3})H_{q_4-1}(X_{i_4})\right]=\sum_{\beta\in\mathscr{D}'_q}C_{q,\beta}\prod_{1\le j<k\le 4}\rho(i_j-i_k)^{\beta_{jk}}, \qquad (7.19)
$$

where $\mathscr{D}'_q$ is the set of elements $(\beta_1,\dots,\beta_6)$, where the β_k's are nonnegative integers satisfying (7.17) and

$$
\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}\ge 1.
$$

Substituting (7.19) into (7.15) yields

$$
\begin{aligned}
\mathrm{Var}\left[\Phi_{n,N}\right]=&\frac{1}{n^2}\sum_{i_1,i_2,i_3,i_4=1}^{n}\sum_{q_1,q_2,q_3,q_4=d}^{N}\sum_{\beta\in\mathscr{D}'_q}C_{q,\beta}q_1q_3c_{q_1}c_{q_2}c_{q_3}c_{q_4} \\
&\times\rho^{\beta_{12}+1}(i_1-i_2)\rho^{\beta_{13}}(i_1-i_3)\rho^{\beta_{14}}(i_1-i_4)\rho^{\beta_{23}}(i_2-i_3)\rho^{\beta_{24}}(i_2-i_4)\rho^{\beta_{34}+1}(i_3-i_4).
\end{aligned}
$$

Replacing $\beta_{12}+1$ and $\beta_{34}+1$ by β_{12} and β_{34}, the above equality can be rewritten as

$$
\mathrm{Var}\left[\Phi_{n,N}\right]=\frac{1}{n^2}\sum_{i_1,i_2,i_3,i_4=1}^{n}\sum_{q_1,q_2,q_3,q_4=d}^{N}\sum_{\beta\in\mathscr{E}_q}K_{q,\beta}c_{q_1}c_{q_2}c_{q_3}c_{q_4}\prod_{1\le j<k\le 4}\rho(i_j-i_k)^{\beta_{jk}},
$$

where

$$
K_{q,\beta}=\frac{q_1!(q_2-1)!q_3!(q_4-1)!}{(\beta_{12}-1)!\beta_{13}!\beta_{14}!\beta_{23}!\beta_{24}!(\beta_{34}-1)!}
$$

and $\mathscr{E}_q$ is the set of nonnegative integers β_{jk}, $1\le j<k\le 4$, satisfying $\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}\ge 1$,

$\beta_{12} \geq 1$, $\beta_{34} \geq 1$ and

$$q_\ell = \sum_{j \,\mathrm{or}\, k = \ell} \beta_{jk}, \quad \text{for} \quad 1 \leq \ell \leq 4.$$

This leads to the estimate

$$\operatorname{Var}[\Phi_{n,N}] \leq \sup_\beta A_{n,\beta} \sum_{q_1,q_2,q_3,q_4=d}^{N} \sum_{\beta \in \mathscr{E}_q} K_{q,\beta} |c_{q_1} c_{q_2} c_{q_3} c_{q_4}|,$$

where

$$A_{n,\beta} = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \prod_{1 \leq j < k \leq 4} |\rho(i_j - i_k)|^{\beta_{jk}},$$

and the supremum is taken over all sets of nonnegative integers β_{jk}, $1 \leq j < k \leq 4$, satisfying $\beta_{13} + \beta_{14} + \beta_{23} + \beta_{24} \geq 1$, $\beta_{12} \geq 1$, $\beta_{34} \geq 1$, $\beta_{jk} \leq d$ for $1 \leq j < k \leq 4$ and

$$d \leq \sum_{j \,\mathrm{or}\, k = \ell} \beta_{jk}, \quad \text{for} \quad 1 \leq \ell \leq 4.$$

To complete the proof we need to show the following claims:

(a) We have
$$\sum_{q_1,q_2,q_3,q_4=d}^{\infty} \sum_{\beta \in \mathscr{E}_q} K_{q,\beta} |c_{q_1} c_{q_2} c_{q_3} c_{q_4}| < \infty. \tag{7.20}$$

(b) If $d = 2$, then $\sup_\beta A_{n,\beta}$ is bounded by a constant times the right-hand side of (7.11).

(c) If $d \geq 3$, then $\sup_\beta A_{n,\beta}$ is bounded by a constant times the right-hand side of (7.12).

Proof of (7.20): The main idea here is to identify the sum in (7.20) as the variance of a truncated function composed with a fixed random variable X_1. From our previous computations it follows that

$$\sum_{q_1,q_2,q_3,q_4=d}^{N} \sum_{\beta\in\mathscr{E}_q} K_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}c_{q_4}| = \sum_{q_1,q_2,q_3,q_4=d}^{N} q_1q_3|c_{q_1}c_{q_2}c_{q_3}c_{q_4}|$$
$$\times \operatorname{Cov}(H_{q_1-1}(X_1)H_{q_2-1}(X_1), H_{q_3-1}(X_1)H_{q_4-1}(X_1))$$
$$= \operatorname{Var}\left[A(g')^{(N)}(X_1)A(g_1)^{(N)}(X_1)\right],$$

where for each integer $N \geq d$, we denote by $A(g')^{(N)}$ and $A(g_1)^{(N)}$ the truncated expansions of $A(g')$ and $A(g_1)$, respectively, introduced in (7.44). By Proposition 7.18, $(A(g')^{(N)})^2$ and $(A(g_1)^{(N)})^2$ are convergent in $L^2(\mathbb{R},\mathscr{B}(\mathbb{R}),\phi(x)dx)$ to $A(g')^2$ and $A(g_1)^2$, respectively. Therefore,

$$\sum_{q_1,q_2,q_3,q_4=d}^{\infty} \sum_{\beta\in\mathscr{E}_q} K_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}c_{q_4}| = \operatorname{Var}\left[A(g')(X_1)A(g_1)(X_1)\right] < \infty.$$

Proof of (b): We will use ideas from graph theory to show the bound in the first part of Theorem 1. Recall the supremum is taken over all sets of nonnegative integers β_{jk}, $1 \leq j < k \leq 4$, satisfying $\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24} \geq 1$, $\beta_{12} \geq 1$, $\beta_{34} \geq 1$, $\beta_{jk} \leq 2$ for $1 \leq j < k \leq 4$ and

$$2 \leq \sum_{j \,\mathrm{or}\, k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \leq \ell \leq 4. \tag{7.21}$$

The exponents β_{jk} induce an unordered simple graph on the set of vertices $V = \{1,2,3,4\}$ by putting an edge between j and k if $\beta_{jk} \neq 0$. There are edges connecting the pairs of vertices $(1,2)$ and $(3,4)$ and condition $\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24} \geq 1$ means that the graph is connected. Without any loss of generality, we can assume that there is an edge between the vertices 2 and 3. Then, condition (7.21) implies that the degree of each vertex is at least two. The worse case is when the number of edges is minimal and the corresponding nonzero coefficients β_{jk} are equal to one. So far we have edges in $(1,2)$, $(3,4)$ and $(2,3)$. There must be more edges because each vertex must have at least degree two. There are two possible cases:

(i) $\beta_{14} = 1$. In this case we have

$$A_{n,\beta} \leq \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} |\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_3-i_4)\rho(i_1-i_4)|.$$

After making the change of variables $i_1 = i_1$, $k_1 = i_1 - i_2$, $k_2 = i_2 - i_3$ and $k_3 = i_3 - i_4$ and using the inequality (A.1) with $M = 3$ and $v = (1,1,1)$, we obtain

$$A_{n,\beta} \leq \frac{1}{n} \sum_{|k_i|\leq n, i=1,2,3} |\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_1+k_2+k_3)| \leq \frac{C}{n}\left(\sum_{|k|\leq n} |\rho(k)|^{\frac{4}{3}}\right)^3.$$

(ii) Suppose that we add two more edges to the graph formed by the edges $(1,2)$, $(2,3)$ and $(3,4)$. In this case, we obtain

$$A_{n,\beta} \leq \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \left|\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_3-i_4)\rho(i_{\alpha_1}-i_{\beta_1})\rho(i_{\alpha_2}-i_{\beta_2})\right|.$$

Making the change of variables $i_1 = i_1$, $k_1 = i_1 - i_2$, $k_2 = i_2 - i_3$ and $k_3 = i_3 - i_4$, we obtain

$$A_{n,\beta} \leq \frac{1}{n} \sum_{|k_i|\leq n, i=1,2,3} |\rho(k_1)\rho(k_2)\rho(k_3)\rho(\mathbf{k}\cdot\mathbf{v})\rho(\mathbf{k}\cdot\mathbf{w})|,$$

where $\mathbf{v}$ and $\mathbf{w}$ are two linearly independent vectors in $\mathbb{Z}^3$ and $\mathbf{k} = (k_1,k_2,k_3)$. Using (A.3), we obtain

$$A_{n,\beta} \leq \frac{C}{n} \sum_{|k|\leq n} |\rho(k)|,$$

which completes the proof of (b).

Proof of (c): This estimate can be obtained by exactly the same arguments as in the proof of Theorem 4.5 in [46]. We omit the details. $\square$

Remark 7.9. We can show that both bounds in (7.11) are not comparable. In the particular case

$|\rho(k)| \sim |k|^{-\alpha}$ as $|k| \to \infty$, with $\alpha > \frac{1}{2}$, we obtain:

$$d_{\mathrm{TV}}(Y_n, Z) \leq \begin{cases} Cn^{1-2\alpha} & \text{if } \quad \frac{1}{2} < \alpha < \frac{2}{3}, \\ Cn^{-\frac{\alpha}{2}} & \text{if } \quad \frac{2}{3} \leq \alpha < 1, \\ Cn^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} & \text{if } \quad \alpha = 1, \\ Cn^{-\frac{1}{2}} & \text{if } \quad \alpha > 1. \end{cases}$$

Theorem 7.10. Assume that $g \in L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ has Hermite rank $d = 2$ and satisfies $A(g) \in \mathbb{D}^{3,8}(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$. Suppose that (7.3) holds true and let Y_n be the random variable defined in (7.2). Then the estimate (7.5) holds true.

Proof. With the notation used in the proof of Theorem 7.8 and using Proposition 3.11, we can write

$$\begin{aligned} \mathrm{d_{TV}}(Y_n, N) &\leq (8 + \sqrt{32\pi})\mathrm{Var}\left[\langle DY_n, u_n/\sigma_n\rangle_{\mathfrak{H}}\right] + \sqrt{2\pi}|\mathrm{E}\left[(Y_n^3)\right]| + \sqrt{32\pi}\mathrm{E}\left[(|D^2_{u_n/\sigma_n}Y_n|^2)\right] \\ &\quad + 4\pi\mathrm{E}\left[|D^3_{u_n/\sigma_n}Y_n|\right] \\ &\leq C\left(\mathrm{Var}\left[\langle DF_n, u_n\rangle_{\mathfrak{H}}\right] + |\mathrm{E}\left[F_n^3\right]| + \mathrm{E}\left[|D^2_{u_n}F_n|^2\right] + \sqrt{\mathrm{E}\left[|D^3_{u_n}F_n|^2)\right]}\right). \end{aligned}$$

Now, we want to estimate each of these terms separately.

Step 1. From Theorem 7.8 we know that

$$\mathrm{Var}(\langle DF_n, u_n\rangle_{\mathfrak{H}} \leq Cn^{-1}\sum_{|k|\leq n}|\rho(k)| + Cn^{-1}\left(\sum_{|k|\leq n}|\rho(k)|^{\frac{4}{3}}\right)^3. \tag{7.22}$$

Step 2. We claim that

$$|\mathrm{E}\left[(F_n^3)\right]| \leq \frac{C}{\sqrt{n}}\left(\sum_{|k|\leq n}|\rho(k)|^{\frac{3}{2}}\right)^2. \tag{7.23}$$

We can write

$$F_n^3 = \frac{1}{n^{3/2}} \sum_{i,j,k=1}^{n} g(X_i)g(X_j)g(X_k).$$

Truncating the Wiener chaos expansion of the random variables $g(X_i)$, as in the proof of Theorem 7.8, we obtain

$$F_n^3 = \lim_{N\to\infty} \Psi_{n,N}^3 := \lim_{N\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{q=2}^{N} c_q H_q(X_i),$$

where the convergence holds in $L^2(\Omega, \mathfrak{F}, \mathrm{P})$ due to Proposition 7.18 because $g \in L^6(\mathbb{R}, \phi(x)dx)$. Therefore,

$$\mathrm{E}\left[(F_n^3)\right] = \lim_{N\to\infty} \mathrm{E}\left[(\Psi_{n,N}^3)\right].$$

We can write

$$\begin{aligned} \mathrm{E}\left[(\Psi_{n,N}^3)\right] =& \frac{1}{n^{3/2}} \sum_{i_1,i_2,i_3=1}^{n} \sum_{q_1,q_2,q_3=2}^{N} c_{q_1}c_{q_2}c_{q_3} \mathrm{E}\left[(H_{q_1}(X_{i_1})H_{q_2}(X_{i_2})H_{q_3}(X_{i_3})\right] \\ =& \frac{1}{n^{3/2}} \sum_{i_1,i_2,i_3=1}^{n} \sum_{q_1,q_2,q_3=2}^{N} c_{q_1}c_{q_2}c_{q_3} \mathrm{E}\left[\left(I_{q_1}(e_{i_1}^{\otimes q_1})I_{q_2}(e_{i_2}^{\otimes q_2})I_{q_3}(e_{i_3}^{\otimes q_3})\right)\right]. \end{aligned} \tag{7.24}$$

Using Lemma 7.17, we obtain

$$\mathrm{E}\left(I_{q_1}(e_{i_1}^{\otimes q_1})I_{q_2}(e_{i_2}^{\otimes q_2})I_{q_3}(e_{i_3}^{\otimes q_3})\right) = \sum_{\beta\in\mathscr{D}_q} C_{q,\beta} \prod_{1\le j<k\le 3} \rho(i_j - i_k)^{\beta_{jk}}, \tag{7.25}$$

where

$$C_{q,\beta} = \frac{\prod_{j=1}^3 q_j!}{\prod_{1\le j<k\le 3} \beta_{jk}!}$$

and $\mathscr{D}_q$ is the set of nonnegative integers β_{jk}, $1 \le j < k \le 3$, satisfying

$$q_\ell = \sum_{j\,\mathrm{or}\,k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \le \ell \le 3. \tag{7.26}$$

Then,

$$|\mathrm{E}\left[(\Psi^3_{n,N})|\right] \le \sup_\beta A_{n,\beta} \sum_{q_1,q_2,q_3=2}^{N} \sum_{\beta\in\mathscr{E}_q} C_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}|,$$

where

$$A_{n,\beta} = \frac{1}{n^{3/2}} \sum_{i_1,i_2,i_3=1}^{n} \prod_{1\le j<k\le 3} |\rho(i_j-i_k)|^{\beta_{jk}},$$

and the supremum is taken over all sets of nonnegative integers β_{jk}, $1\le j<k\le 3$, satisfying $\beta_{jk}\le 2$ for $1\le j<k\le 3$ and

$$2 \le \sum_{j\,\mathrm{or}\,k=\ell} \beta_{jk}, \quad \text{for} \quad 1\le \ell \le 3.$$

It is easy to see that to satisfy the above conditions, $\beta_{jk}\ge 1$ for all $1\le j<k\le 3$. Hence, we have

$$A_{n,\beta} \le \frac{1}{n^{3/2}} \sum_{i_1,i_2,i_3=1}^{n} |\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_3)|.$$

After making the change of variables $i_1=i_1$, $k_1=i_1-i_2$, $k_2=i_1-i_3$ and using the inequality (A.1) with $M=2$ and $v=(-1,1)$, we obtain

$$A_{n,\beta} \le \frac{1}{n^{1/2}} \sum_{|k_1|,|k_2|\le n} |\rho(k_1)\rho(k_2)\rho(k_2-k_1)| \le \frac{C}{\sqrt{n}} \left(\sum_{|k|\le n} |\rho(k)|^{\frac{3}{2}}\right)^2.$$

To complete the proof of (7.23), we need to show that:

$$\sum_{q_1,q_2,q_3=2}^{\infty} \sum_{\beta\in\mathscr{D}_q} C_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}| < \infty.$$

In fact,

$$\lim_{N\to\infty} \sum_{q_1,q_2,q_3=2}^{N} \sum_{\beta\in\mathscr{D}_q} C_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}| = \lim_{N\to\infty} \mathrm{E}\left[\left(A(g)^N\right)^3)\right] = \mathrm{E}\left[\left((A(g))^3\right)\right] < \infty,$$

taking into account Proposition 7.18 and the fact that $A(g)\in L^6(\mathbb{R},\phi(x)dx)$.

Step 3. We proceed now with the estimation of $\mathrm{E}\left[(|D^2_{u_n}F_n|^2)\right]$. We can write

$$D_{u_n}F_n = \langle DF_n, u_n\rangle_{\mathfrak{H}} = \frac{1}{n}\sum_{i,j=1}^{n} g'(X_i)g_1(X_j)\rho(i-j)$$

and

$$D(\langle DF_n, u_n\rangle_{\mathfrak{H}}) = \frac{1}{n}\sum_{i,j=1}^{n}(g''(X_i)g_1(X_j)e_i + g'(X_i)g_1'(X_j)e_j)\rho(i-j).$$

Therefore,

$$\begin{aligned} D^2_{u_n}F_n &= \langle u_n, D(\langle DF_n, u_n\rangle_{\mathfrak{H}})\rangle_{\mathfrak{H}} \\ &= \frac{1}{n^{3/2}}\sum_{i,j,k=1}^{n}(g''(X_i)g_1(X_j)g_1(X_k)\rho(i-k) + g'(X_i)g_1'(X_j)g_1(X_k)\rho(j-k))\rho(i-j). \end{aligned} \tag{7.27}$$

Because the random variables $g''(X_i)$, $g_1(X_j)$, $g_1(X_k)$, $g'(X_i)$ and $g_1'(X_j)$ appearing in the above expression belong to $L^2(\Omega)$, their truncated Wiener chaos expansions convergence in $L^2(\Omega)$, and, as a consequence, $D^2_{u_n}F_n = \lim_{N\to\infty}\Phi_{n,N}$ in probability, where

$$\begin{aligned} \Phi_{n,N} = \frac{1}{n^{3/2}}\sum_{i_1,i_2,i_3=1}^{n}\sum_{q_1,q_2,q_3=2}^{N} & c_{q_1}c_{q_2}c_{q_3}q_1(q_1-1)H_{q_1-2}(X_{i_1})H_{q_2-1}(X_{i_2})H_{q_3-1}(X_{i_3}) \\ & \times\rho(i_1-i_2)\rho(i_1-i_3) \\ & + c_{q_1}c_{q_2}c_{q_3}q_1(q_2-1)H_{q_1-1}(X_{i_1})H_{q_2-2}(X_{i_2})H_{q_3-1}(X_{i_3})\rho(i_1-i_2)\rho(i_2-i_3). \end{aligned}$$

Making the change of variables $(q_1,q_2)\to(q_2,q_1)$ and $(i_1,i_2)\to(i_2,i_1)$ in the second sum allows us to put the two terms together, and we obtain

$$\begin{aligned} \Phi_{n,N} = \frac{1}{n^{3/2}}\sum_{i_1,i_2,i_3=1}^{n}\sum_{q_1,q_2,q_3=2}^{N} & c_{q_1}c_{q_2}c_{q_3}(q_1+q_2)(q_1-1)H_{q_1-2}(X_{i_1})H_{q_2-1}(X_{i_2})H_{q_3-1}(X_{i_3}) \\ & \times\rho(i_1-i_2)\rho(i_1-i_3). \end{aligned}$$

Therefore, by Fatou's lemma,

$$\mathrm{E}\left[|D^2_{u_n}F_n|^2\right] \le \liminf_{N\to\infty} \mathrm{E}\left[|\Phi^2_{n,N}|\right].$$

Then,

$$|\Phi_{n,N}|^2 = \frac{1}{n^3}\sum_{i_1,\dots,i_6=1}^{n}\sum_{q_1,\dots,q_6=2}^{N} C_q H_{q_1-2}(X_{i_1})H_{q_2-1}(X_{i_2})H_{q_3-1}(X_{i_3})$$
$$\times H_{q_4-2}(X_{i_4})H_{q_5-1}(X_{i_5})H_{q_6-1}(X_{i_6})\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_4-i_5)\rho(i_4-i_6),$$

where

$$C_q = c_{q_1}c_{q_2}c_{q_3}c_{q_4}c_{q_5}c_{q_6}(q_1+q_2)(q_1-1)(q_4+q_5)(q_4-1).$$

Using the product formula for multiple integrals (see Lemma 7.17), we get

$$\mathrm{E}\left[\left(|\Phi_{n,N}|^2\right)\right] = \frac{1}{n^3}\sum_{i_1,\dots,i_6=1}^{n}\sum_{q_1,\dots,q_6=2}^{N}\sum_{\beta\in\mathscr{D}_q} K_{q,\beta}\left(\prod_{1\le k<l\le 6}\rho(i_k-i_l)^{\beta_{kl}}\right)$$
$$\times\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_4-i_5)\rho(i_4-i_6),$$

where

$$K_{q,\beta} = \frac{(q_1+q_2)(q_4+q_5)\prod_{j=1}^{6}c_{q_j}(q_j-1)!}{\prod_{1\le k<l\le 6}\beta_{kl}!}$$

and

$$\mathscr{D}_q = \{(\beta_{kl})_{1\le k<l\le 6} : \sum_{k\text{ or }l=j}\beta_{kl} = q_j-1 \text{ for } j=2,3,5,6 \text{ and } \sum_{k\text{ or }l=j}\beta_{kl} = q_j-2 \text{ for } j=1,4\}.$$

Replacing $\beta_{jk}+1$ by β_{jk} for $(j,k)\in\{(1,2),(1,3),(4,5),(4,6)\}$, yields

$$\mathrm{E}\left[\left(|\psi_{n,N}|^2\right)\right] = \frac{1}{n^3}\sum_{i_1,\dots,i_6=1}^{n}\sum_{q_1,\dots,q_6=2}^{N}\sum_{\beta\in\mathscr{C}_q} L_{q,\beta}\left(\prod_{1\le k<l\le 6}\rho(i_k-i_l)^{\beta_{kl}}\right),$$

where

$$L_{q,\beta} = \frac{(q_1+q_2)(q_4+q_5)\prod_{i=1}^6 c_{q_i}(q_i-1)!}{(\beta_{12}+1)!(\beta_{13}+1)!\beta_{14}!\beta_{15}!\beta_{16}!\beta_{23}!\beta_{24}!\beta_{25}!\beta_{26}!\beta_{34}!\beta_{35}!\beta_{36}!(\beta_{45}+1)!(\beta_{46}+1)!\beta_{56}!}$$

and

$$\mathscr{C}_q = \{(\beta_{kl})_{1\le k<l\le 6} : \sum_{k \text{ or } l=j} \beta_{kl} = q_j \text{ for } j=1,\dots,6 \text{ and } \beta_{12}, \beta_{13}, \beta_{45}. \beta_{46} \ge 1\}.$$

Then, we can write

$$\mathrm{E}\left[\left(|\psi_{n,N})|^2\right)\right] \le \sup_{\beta\in\mathscr{C}_q} A_{n,\beta} \sum_{q_1,\dots,q_6=2}^{N} \sum_{\beta\in\mathscr{C}_q} |L_{q,\beta}|,$$

where

$$A_{n,\beta} = \frac{1}{n^3} \sum_{i_1,i_2,i_3,i_4=1}^{n} \prod_{1\le j<k\le 6} |\rho(i_i - i_k)|^{\beta_{jk}}$$

and the supremum is taken over all sets of nonnegative integers β_{jk}, $1\le j<k\le 6$, satisfying $\beta_{12}, \beta_{13}, \beta_{45}, \beta_{46} \ge 1$, $\beta_{jk} \le 2$ for $1\le j<k\le 6$ and

$$2 \le \sum_{j \,\mathrm{or}\, k=\ell} \beta_{jk}, \quad \text{for} \quad 1\le \ell \le 6.$$

Then, the estimation follows as in the proof of the last part of Theorem 7.15.

Now, we need to show that

$$\sum_{q_1,\dots,q_6=2}^{\infty} \sum_{\beta\in\mathscr{C}_q} |L_{q,\beta}| < \infty. \tag{7.28}$$

In fact,

$$\begin{aligned}
\sum_{q_1,\dots,q_6=2}^{N} \sum_{\beta\in\mathscr{C}_q} |L_{q,\beta}| &= \sum_{q_1,\dots,q_6=2}^{N} \left(\prod_{i=1}^{6} |c_{q_i}|\right)(q_1+q_2)(q_1-1)(q_3+q_4)(q_4-1) \\
&\quad \times \mathrm{E}\left[\left(H_{q_1-2}(X_1)H_{q_2-1}(X_1)H_{q_3-1}(X_1)H_{q_4-2}(X_1)H_{q_5-1}(X_1)H_{q_6-1}(X_1)\right)\right] \\
&= \mathrm{E}\left[\left(A(g'')^{(N)})^2(A(g_1)^{(N)})^4\right)\right] \le \|A(g'')^{(N)}\|_{L^6(\mathbb{R},\phi(x)dx)}^{\frac{1}{3}} \|A(g_1)^{(N)}\|_{L^6(\mathbb{R},\phi(x)dx)}^{\frac{2}{3}}.
\end{aligned}$$

Since $A(g) \in \mathbb{D}^{3,6}$, $(A(g'')^{(N)})^3$ and $(A(g_1)^{(N)})^3$ converge to $A(g'')$ and $A(g_1)$, respectively, in $L^2(\mathbb{R}, \phi(x)dx)$ by Proposition 7.18. Then, (7.28) is true.

Step 4. We proceed to the estimation of $\sqrt{\mathrm{E}\left[(|D^3_{u_n}F_n|^2)\right]}$. Taking the derivative in (7.27), yields

$$
\begin{aligned}
D(D^2_{u_n}F_n) = \frac{1}{n^{3/2}} \sum_{i,j,k=1}^{n} & g'''(X_i)g_1(X_j)g_1(X_k)\rho(i-j)\rho(i-k)e_i \\
& + g''(X_i)g_1'(X_j)g_1(X_k)\rho(i-j)\rho(i-k)e_j + g''(X_i)g_1(X_j)g_1'(X_k)\rho(i-j)\rho(i-k)e_k \\
& + g''(X_i)g_1'(X_j)g_1(X_k)\rho(i-j)\rho(j-k)e_i + g'(X_i)g_1''(X_j)g_1(X_k)\rho(i-j)\rho(j-k)e_j \\
& + g'(X_i)g_1'(X_j)g_1'(X_k)\rho(i-j)\rho(j-k)e_k.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\langle u_n, D(D^2_{u_n}F_n\rangle_{\mathfrak{H}} = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} & g'''(X_{i_1})g_1(X_{i_2})g_1(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_1-i_4) \\
& + g''(X_{i_1})g_1'(X_{i_2})g_1(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4) \\
& + g''(X_{i_1})g_1(X_{i_2})g_1'(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_3-i_4) \\
& + g''(X_{i_1})g_1'(X_{i_2})g_1(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_1-i_4) \\
& + g'(X_{i_1})g_1''(X_{i_2})g_1(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_2-i_4) \\
& + g'(X_{i_1})g_1'(X_{i_2})g_1'(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_3-i_4).
\end{aligned}
$$

Notice that the second, third and fourth terms are identical. This allows us to write

$$
\begin{aligned}
D^3_{u_n}F_n = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} & g'''(X_{i_1})g_1(X_{i_2})g_1(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_1-i_4) \\
& + 3g''(X_{i_1})g_1'(X_{i_2})g_1(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4) \\
& + g'(X_{i_1})g_1''(X_{i_2})g_1(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_2-i_4) \\
& + g'(X_{i_1})g_1'(X_{i_2})g_1'(X_{i_3})g_1(X_{i_4})\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_3-i_4).
\end{aligned}
$$

Then, we have

$$D^3_{u_n} V_n = \lim_{N\to\infty} \Phi_{n,N},$$

where the convergence holds in probability and

$$\begin{aligned}
\Phi_{n,N} = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \sum_{q_1,q_2,q_3,q_4=2}^{N} & C_q^{(1)} H_{q_1-3}(X_{i_1}) H_{q_2-1}(X_{i_2}) H_{q_3-1}(X_{i_3}) H_{q_4-1}(X_{i_4}) \\
& \times \rho(i_1-i_2)\rho(i_1-i_3)\rho(i_1-i_4) \\
& + C_q^{(2)} H_{q_1-2}(X_{i_1}) H_{q_2-2}(X_{i_2}) H_{q_3-1}(X_{i_3}) H_{q_4-1}(X_{i_4}) \rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4) \\
& + C_q^{(3)} H_{q_1-1}(X_{i_1}) H_{q_2-3}(X_{i_2}) H_{q_3-1}(X_{i_3}) H_{q_4-1}(X_{i_4}) \rho(i_1-i_2)\rho(i_2-i_3)\rho(i_2-i_4) \\
& + C_q^{(4)} H_{q_1-1}(X_{i_1}) H_{q_2-2}(X_{i_2}) H_{q_3-2}(X_{i_3}) H_{q_4-1}(X_{i_4}) \rho(i_1-i_2)\rho(i_2-i_3)\rho(i_1-i_4)
\end{aligned}$$

with

$$\begin{aligned}
C_q^{(1)} &= c_{q_1} c_{q_2} c_{q_3} c_{q_4} q_1 (q_1-1)(q_1-2), \\
C_q^{(2)} &= 3 c_{q_1} c_{q_2} c_{q_3} c_{q_4} q_1 (q_1-1)(q_2-1), \\
C_q^{(3)} &= c_{q_1} c_{q_2} c_{q_3} c_{q_4} q_1 (q_2-1)(q_2-2), \\
C_q^{(4)} &= c_{q_1} c_{q_2} c_{q_3} c_{q_4} q_1 (q_2-1)(q_3-1).
\end{aligned}$$

We can combine the first and third terms with the change of variables $(q_1,q_2) \to (q_2,q_1)$ and $(i_1,i_2) \to (i_2,i_1)$. In this way we obtain

$$\begin{aligned}
\Phi_{n,N} = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \sum_{q_1,q_2,q_3,q_4=2}^{N} & \widetilde{C}_q^{(1)} H_{q_1-3}(X_{i_1}) H_{q_2-1}(X_{i_2}) H_{q_3-1}(X_{i_3}) H_{q_4-1}(X_{i_4}) \\
& \times \rho(i_1-i_2)\rho(i_1-i_3)\rho(i_1-i_4) \\
& + \widetilde{C}_q^{(2)} H_{q_1-2}(X_{i_1}) H_{q_2-2}(X_{i_2}) H_{q_3-1}(X_{i_3}) H_{q_4-1}(X_{i_4}) \rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4) \\
& + \widetilde{C}_q^{(3)} H_{q_1-1}(X_{i_1}) H_{q_2-2}(X_{i_2}) H_{q_3-2}(X_{i_3}) H_{q_4-1}(X_{i_4}) \rho(i_1-i_2)\rho(i_2-i_3)\rho(i_1-i_4) \\
=: \Phi_{n,N}^{(1)} + \Phi_{n.N}^{(2)} + \Phi_{n.N}^{(3)}
\end{aligned}$$

with

$$\widetilde{C}_q^{(1)} = c_{q_1}c_{q_2}c_{q_3}c_{q_4}(q_1+q_2)(q_1-1)(q_1-2),$$
$$\widetilde{C}_q^{(2)} = c_{q_1}c_{q_2}c_{q_3}c_{q_4}3q_1(q_1-1)(q_2-1),$$
$$\widetilde{C}_q^{(3)} = c_{q_1}c_{q_2}c_{q_3}c_{q_4}q_1(q_2-1)(q_3-1).$$

Then, by Fatou's lemma,

$$\mathrm{E}\left[\left(|D^3_{u_n}V_n|^2\right)\right] \le \liminf_{N\to\infty}\mathrm{E}\left[\left(|\Phi_{n,N}|^2\right)\right].$$

We are going to treat each term $\Phi^{(i)}_{n,N}$, $i=1,2,3$, separately.

Case $i=1$. Let us first estimate $\mathrm{E}\left[\left(|\Phi^{(1)}_{n,N}|^2\right)\right]$. We have

$$\begin{aligned}\mathrm{E}\left((\Phi^{(1)}_{n,N})^2\right) = \frac{1}{n^4}\sum_{i_1,\dots,i_8=1}^{n}\sum_{q_1,\dots,q_8=2}^{N} M_q^{(1)}\mathrm{E}\big(&H_{q_1-3}(X_{i_1})H_{q_2-1}(X_{i_2})H_{q_3-1}(X_{i_3})H_{q_4-1}(X_{i_4})\\ &\times H_{q_5-3}(X_{i_5})H_{q_6-1}(X_{i_6})H_{q_7-1}(X_{i_7})H_{q_8-1}(X_{i_8})\big)\\ &\times\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_1-i_4)\rho(i_5-i_6)\rho(i_5-i_7)\rho(i_5-i_8),\end{aligned}$$

where

$$M_q^{(1)} = \left(\prod_{j=1}^{8}c_{q_j}\right)(q_1+q_2)(q_1-1)(q_1-2)(q_5+q_6)(q_5-1)(q_5-2).$$

This yields

$$\begin{aligned}\mathrm{E}\left[\Phi^{(1)}_{n,N})^2\right] \le \frac{1}{n^4}\sum_{i_1,\dots,i_8=1}^{n}\sum_{q_1,\dots q_8=2}^{N}\sum_{\beta\in\mathscr{D}_q^{(1)}}&K^{(1)}_{q,\beta}\left(\prod_{1\le k<l\le 8}|\rho(i_k-i_l)|^{\beta_{kl}}\right)\\ &\times|\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_1-i_4)\rho(i_5-i_6)\rho(i_5-i_7)\rho(i_5-i_8)|,\end{aligned}$$

where

$$K^{(1)}_{q,\beta} = \frac{(q_1+q_2)(q_5+q_6)\prod_{j=1}^{8}|c_{q_j}|(q_j-1)!}{\prod_{1\le k<l\le 8}\beta_{kl}!},$$

and

$$\mathscr{D}_q^{(1)} = \{(\beta_{kl})_{1\le k<l\le 8} : \sum_{k\ or\ l=j} \beta_{kl} = q_j - 1 \text{ for } j = 2,3,4,6,7,8$$
$$\text{and} \sum_{k\ or\ l=j} \beta_{kl} = q_j - 3 \text{ for } j = 1,5\}.$$

Changing the exponents $\beta_{jk}+1$ in to β_{jk} for $(j,k) \in \{(1,2),(1,3),(1,4),(5,6),(5,7),(5,8)\}$, we can write

$$\mathrm{E}\left[\right](\Phi_{n,N}^{(1)})^2\Big] \le \frac{1}{n^4} \sum_{i_1,\dots,i_8=1}^{n} \sum_{q_1,\dots q_8=2}^{N} \sum_{\beta\in\mathscr{C}_q^{(1)}} L_{q,\beta}^{(1)} \left(\prod_{1\le k<l\le 8} |\rho(i_k-i_l)|^{\beta_{kl}} \right),$$

where

$$L_{q,\beta}^{(1)} = \frac{(q_1+q_))(q_5+q_6)\prod_{j=1}^{8}|c_{q_j}|(q_j-1)!}{(\beta_{12}-1)!(\beta_{13}-1)!(\beta_{14}-1)!(\beta_{56}-1)!(\beta_{57}-1)(\beta_{58}-1)!\prod_{(k,l)\in\mathscr{E}}\beta_{kl}!},$$

with $\mathscr{E} = \{(k,l) : 1 \le k < l \le 8, (k,l) \ne (1,2),(1,3),(1,4),(5,6),(5,7),(5,8)\}$ and

$$\mathscr{C}_q^{(1)} = \{(\beta_{kl})_{1\le k<l\le 8} : \sum_{k\ or\ l=j} \beta_{kl} = q_j \text{ for } j = 1,\dots,8 \text{ and } \beta_{12}, \beta_{13}, \beta_{14}, \beta_{56}, \beta_{57}, \beta_{58} \ge 1\}.$$

Then, we obtain

$$\mathrm{E}\left[\right](\Phi_{n,N}^{(1)})^2\Big] \le \sup_{\beta\in\mathscr{C}_q^{(11)}} A_{n,\beta}^{(1)} \sum_{q_1,\dots,q_8=2}^{N} \sum_{\beta\in\mathscr{C}_q^{(1)}} |L_{q,\beta}^{(1)}|,$$

where

$$A_{n,\beta}^{(1)} = \frac{1}{n^4} \sum_{i_1,\dots,i_8=1}^{n} \prod_{1\le j\le k\le 8} |\rho(i_i-i_k)|^{\beta_{jk}}$$

and the supremum is taken over all sets of nonnegative integers β_{jk}, $1 \le j < k \le 8$, satisfying $\beta_{12}, \beta_{13}, \beta_{14}, \beta_{56}, \beta_{57}, \beta_{58} \ge 1$, $\beta_{jk} \le 2$ for $1 \le j < k \le 8$ and

$$2 \le \sum_{j\,\mathrm{or}\,k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \le \ell \le 8.$$

We need to estimate $A_{n,\beta}^{(1)}$ and to show that

$$\sum_{q_1,\dots,q_8=2}^{\infty} \sum_{\beta \in \mathscr{C}_q^{(1)}} L_{q,\beta}^{(1)} < \infty. \tag{7.29}$$

Estimation of $A_{n,\beta}^{(1)}$: We claim that

$$\sup_{\beta} A_{n,\beta}^{(1)} \leq C n^{-1} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{3}{2}} \right)^4. \tag{7.30}$$

As in the proof of Theorem 7.15, we will make use of ideas from graph theory. The exponents β_{jk} induce an unordered simple graph on the set of vertices $V = \{1,2,3,4,5,8\}$ by putting an edge between j and k whenever $\beta_{jk} \neq 0$. Because $\beta_{12}, \beta_{13} \geq 1$, $\beta_{14} \geq 1$, $\beta_{56} \geq 1, \beta_{57} \geq 1$ and $\beta_{58} \geq 1$, there are edges connecting the pairs of vertices $(1,2)$, $(1,3)$, $(1,4)$, $(5,6)$, $(5,7)$ and $(5,8)$. Condition (7.41) means that the degree of each vertex is at least 2. Then we consider two cases, depending whether graph is connected or not.

Case 1: Suppose that the graph is not connected. This means that $\beta_{jk} = 0$ if $j \in \{1,2,3,4\}$ and $k \in \{5,6,7,8\}$ and there is no edge between the sets $V_1 = \{1,2,3,4\}$ and $V_2 = \{5,6,7,8\}$. Therefore,

$$A_{n,\beta}^{(1)} \leq (A_{n,\beta}^{(0)})^2,$$

where

$$A_{n,\beta}^{(0)} = \frac{1}{n^2} \sum_{i_1,\dots,i_4=1}^{n} \prod_{1 \leq j \leq k \leq 4} |\rho(i_i - i_k)|^{\beta_{jk}}$$

and the nonnegative integers β_{jk}, $1 \leq j < k \leq 4$, satisfy β_{12}, β_{13}, $\beta_{14} \geq 1$, $\beta_{jk} \leq 2$ for $1 \leq j < k \leq 4$ and

$$2 \leq \sum_{j \,\mathrm{or}\, k = \ell} \beta_{jk}, \quad \text{for} \quad 1 \leq \ell \leq 4.$$

As a consequence, $\beta_{23} + \beta_{24} \geq 1$, $\beta_{23} + \beta_{34} \geq 1$ and $\beta_{24} + \beta_{34} \geq 1$. This means that at least two of the indices β_{23}, β_{24} and β_{34} is larger or equal to 1. Considering the worst case, we can assume that

$\beta_{23} = 1$ and $\beta_{34} = 1$. This leads to

$$A_{n,\beta}^{(0)} \leq n^{-1} \sum_{|k_1|,|k_2|,|k_3| \leq n} |\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_2-k_1)\rho(k_3-k_2)|. \tag{7.31}$$

Using (A.3) and Hölder's inequality we obtain

$$A_{n,\beta}^{(0)} \leq Cn^{-1} \sum_{|k| \leq n} |\rho(k)| \leq Cn^{-\frac{2}{3}} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{3}{2}} \right)^{\frac{2}{3}}.$$

Case 2: Suppose that the graph is connected. This means that there is an edge connecting the sets V_1 and V_2. Suppose that $\beta_{\alpha_0 \delta_0} \geq 1$, where $\alpha_0 \in \{1,2,3,4\}$ and $\delta_0 \in \{5,6,7,8\}$. We have then 7 nonzero coefficients β: β_{13}, β_{13}, β_{14}, β_{56}, β_{57}, β_{58} and $\beta_{\alpha_0 \delta_0}$. Because all the edges have at least degree 2, there must be another nonzero coefficient β. Assume it is $\beta_{\alpha_1 \delta_1}$. Then, the worse case will be when $\beta_{12} = \beta_{13} = \beta_{14} = \beta_{56} = \beta_{57} = \beta_{58} = \beta_{\alpha_0 \delta_0} = \beta_{\alpha_1 \delta_1} = 1$ and all the other coefficients are zero. Consider the change of variables $i_1 - i_2 = k_1$, $i_1 - i_3 = k_2$, $i_1 - i_4 = k_3$, $i_5 - i_6 = k_4$, $i_5 - i_7 = k_5$, $i_5 - i_8 = k_6$, $i_{\alpha_0} - i_{\delta_0} = k_7$. Then, it is easy to show that $i_{\alpha_1} - i_{\delta_1} = \mathbf{k} \cdot \mathbf{v}$, where $\mathbf{k} = (k_1, \ldots, k_5)$ and $\mathbf{v}$ is a 7-dimensional vector whose components are 0, 1 or -1. Applying (A.2) and Hölder's inequality yields

$$A_{n,\beta}^{(1)} \leq Cn^{-2} \left(\sum_{|k| \leq n} |\rho(k)| \right)^6 \leq Cn^{-2} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{3}{2}} \right)^4.$$

This completes the proof of (7.30).

Proof of (7.29): We have

$$\sum_{q_1,\ldots,q_8=2}^{\infty} \sum_{\beta \in \mathscr{C}_q^{(1)}} L_{q,\beta}^{(1)} = \mathrm{E}\Big(\Big| (A(g''')^{(N)})(X_1)(A(g_1)^{(N)}(X_1))^3$$
$$+ (A(g')^{(N)})(X_1)(A(g'')^{(N)})(X_1)(A(g_1)^{(N)}(X_1))^2 \Big|^2 \Big).$$

Applying Hölder's inequality, yields

$$\sum_{q_1,\dots,q_8=2}^{\infty} \sum_{\beta \in \mathscr{C}_q^{(1)}} L_{q,\beta}^{(1)} \leq 2\|A(g''')^{(N)}\|_{L^8(\mathbb{R},\phi(x)dx)}^2 \|A(g_1)^{(N)}\|_{L^8(\mathbb{R},\phi(x)dx)}^6$$
$$+2\|A(g')^{(N)}\|_{L^8(\mathbb{R},\phi(x)dx)}^2 \|A(g'')^{(N)}\|_{L^8(\mathbb{R},\phi(x)dx)}^2 \|A(g_1)^{(N)}\|_{L^8(\mathbb{R},\phi(x)dx)}^4.$$

By Equation 7.44 and our hypothesis, taking the limit as N tends to infinity, it follows that

$$\sum_{q_1,\dots,q_8=2}^{\infty} \sum_{\beta \in \mathscr{C}_q^{(1)}} L_{q,\beta}^{(1)} \leq 2\|A(g''')\|_{L^8(\mathbb{R},\phi(x)dx)}^2 \|A(g_1)\|_{L^8(\mathbb{R},\phi(x)dx)}^6$$
$$+2\|A(g')\|_{L^8(\mathbb{R},\phi(x)dx)}^2 \|A(g'')\|_{L^8(\mathbb{R},\phi(x)dx)}^2 \|A(g_1)\|_{L^8(\mathbb{R},\phi(x)dx)}^4 < \infty.$$

Case $i = 2$. For $\mathrm{E}\left[[|\Phi_{n,N}^{(2)}|^2]\right]$ we have

$$\mathrm{E}\left((\Phi_{n,N}^{(2)})^2\right) = \frac{1}{n^4} \sum_{i_1,\dots,i_8=1}^{n} \sum_{q_1,\dots,q_8=2}^{N} M_q^{(2)} \mathrm{E}\left(H_{q_1-2}(X_{i_1})H_{q_2-2}(X_{i_2})H_{q_3-1}(X_{i_3})H_{q_4-1}(X_{i_4})\right.$$
$$\left.\times H_{q_5-2}(X_{i_5})H_{q_6-2}(X_{i_6})H_{q_7-1}(X_{i_7})H_{q_8-1}(X_{i_8})\right)$$
$$\times \rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4)\rho(i_5-i_6)\rho(i_5-i_7)\rho(i_6-i_8),$$

where

$$M_q^{(2)} = \left(\prod_{j=1}^{8} c_{q_j}\right) 9q_1(q_1-1)(q_2-1)q_5(q_5-1)(q_6-1)).$$

This yields

$$\mathrm{E}\left((\Phi_{n,N}^{(2)})^2\right) \leq \frac{1}{n^4} \sum_{i_1,\dots,i_8=1}^{n} \sum_{q_1,\dots q_8=2}^{N} \sum_{\beta \in \mathscr{D}_q^{(2)}} K_{q,\beta}^{(2)} \left(\prod_{1\leq k<l\leq 8} |\rho(i_k-i_l)|^{\beta_{kl}}\right)$$
$$\times |\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4)\rho(i_5-i_6)\rho(i_5-i_7)\rho(i_6-i_8)|,$$

where

$$K_{q,\beta}^{(2)} = \frac{9q_1q_5\prod_{j=1}^{8}|c_{q_j}|(q_j-1)!}{\prod_{1\leq k<l\leq 8}\beta_{kl}!}$$

and

$$\mathscr{D}_q^{(2)} = \{(\beta_{kl})_{1\le k<l\le 8} : \sum_{k \ or \ l=j} \beta_{kl} = q_j - 1 \text{ for } j = 3,4,7,8$$
$$\text{and} \sum_{k \ or \ l=j} \beta_{kl} = q_j - 2 \text{ for } j = 1,2,5,6\}.$$

Changing the exponents $\beta_{jk}+1$ in to β_{jk} for $(j,k) \in \{(1,2),(1,3),(2,4),(5,6),(5,7),(6,8)\}$, we can write

$$\mathrm{E}\left((\Phi_{n,N}^{(2)})^2\right) \le \frac{1}{n^4} \sum_{i_1,\dots,i_8=1}^{n} \sum_{q_1,\dots q_8=2}^{N} \sum_{\beta \in \mathscr{C}_q^{(2)}} L_{q,\beta}^{(2)} \left(\prod_{1\le k<l\le 8} |\rho(i_k - i_l)|^{\beta_{kl}} \right),$$

where

$$L_{q,\beta}^{(2)} = \frac{9 q_1 q_5 \prod_{j=1}^{8} |c_{q_j}|(q_j-1)!}{(\beta_{12}-1)!(\beta_{13}-1)!(\beta_{24}-1)!(\beta_{56}-1)!(\beta_{57}-1)(\beta_{68}-1)!\prod_{(k,l)\in\mathscr{E}} \beta_{kl}!}\ ,$$

with $\mathscr{E} = \{(k,l) : 1 \le k < l \le 8, (k,l) \ne (1,2),(1,3),(2,4),(5,6),(5,7),(6,8)\}$ and

$$\mathscr{C}_q^{(2)} = \{(\beta_{kl})_{1\le k<l\le 8} : \sum_{k \ or \ l=j} \beta_{kl} = q_j \text{ for } j = 1,\dots,8 \text{ and } \beta_{12}, \beta_{13}, \beta_{24}, \beta_{56}, \beta_{57}, \beta_{6,8} \ge 1\}.$$

Then, we have

$$\mathrm{E}\left(\Phi_{n,N}^{(2)})^2\right) \le \sup_{\beta \in \mathscr{C}_q^{(12)}} A_{n,\beta}^{(2)} \sum_{q_1,\dots,q_8=2}^{N} \sum_{\beta\in\mathscr{C}_q^{(2)}} |L_{q,\beta}^{(2)}|,$$

where

$$A_{n,\beta}^{(2)} = \frac{1}{n^4} \sum_{i_1,\dots,i_8=1}^{n} \prod_{1\le j\le k\le 8} |\rho(i_i - i_k)|^{\beta_{jk}}$$

and the supremum is taken over all sets of nonnegative integers β_{jk}, $1 \le j < k \le 8$, satisfying $\beta_{12}, \beta_{13}, \beta_{24}, \beta_{56}, \beta_{57}, \beta_{68} \ge 1$, $\beta_{jk} \le 2$ for $1 \le j < k \le 8$ and

$$2 \le \sum_{j \,\mathrm{or}\, k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \le \ell \le 8.$$

We need to estimate $A^{(2)}_{n,\beta}$ and to show that

$$\sum_{q_1,\ldots,q_8=2}^{\infty} \sum_{\beta\in\mathscr{C}_q^{(2)}} L^{(2)}_{q,\beta} < \infty. \tag{7.32}$$

Estimation of $A^{(2)}_{n,\beta}$: We claim that

$$\sup_\beta A^{(2)}_{n,\beta} \le Cn^{-1}\left(\sum_{|k|\le n} |\rho(k)|^{\frac{3}{2}}\right)^4.$$

As in the proof of Theorem 7.15, we will make use of ideas from graph theory. The exponents β_{jk} induce an unordered simple graph on the set of vertices $V = \{1,2,3,4,5,8\}$ by putting an edge between j and k whenever $\beta_{jk} \neq 0$. Because $\beta_{12} \ge 1$, $\beta_{13} \ge 1$, $\beta_{24} \ge 1$, $\beta_{56} \ge 1, \beta_{57} \ge 1$ and $\beta_{68} \ge 1$, there are edges connecting the pairs of vertices $(1,2)$, $(1,3)$, $(2,4)$, $(5,6)$ $(5,7)$ and $(6,8)$. Condition (7.41) means that the degree of each vertex is at least 2. Then we consider two cases, depending whether graph is connected or not.

Case 1: Suppose that the graph is not connected. This means that $\beta_{jk} = 0$ if $j \in \{1,2,3,4\}$ and $k \in \{5,6,7,8\}$ and there is no edge between the sets $V_1 = \{1,2,3,4\}$ and $V_2 = \{5,6,7,8\}$. Therefore,

$$A^{(2)}_{n,\beta} \le (A^{(0)}_{n,\beta})^2,$$

where

$$A^{(0)}_{n,\beta} = \frac{1}{n^2} \sum_{i_1,\ldots,i_4=1}^{n} \prod_{1\le j\le k\le 4} |\rho(i_i - i_k)|^{\beta_{jk}}$$

and the nonnegative integers β_{jk}, $1 \le j < k \le 4$, satisfy $\beta_{12}, \beta_{13}, \beta_{24} \ge 1$, $\beta_{jk} \le 2$ for $1 \le j < k \le 4$ and

$$2 \le \sum_{j \,\mathrm{or}\, k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \le \ell \le 4.$$

As a consequence, $\beta_{23} + \beta_{34} \ge 1$ and $\beta_{14} + \beta_{34} \ge 1$. This means $\beta_{34} \ge 1$ or both β_{23} and β_{14} are larger or equal than one. There are two possible cases:

(i) Suppose $\beta_{34} \geq 1$, Considering the worst case, we can assume that $\beta_{34} = 1$. Then, applying (A.1) and Hölder's inequality, we obtain

$$A_{n,\beta}^{(0)} \leq n^{-1} \sum_{|k_1|,|k_2|,|k_3| \leq n} |\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_1+k_3-k_2)| \leq n^{-1} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{4}{3}} \right)^3.$$

By Hölder's inequality, we can show that

$$(A_{n,\beta}^{(0)})^2 \leq Cn^{-1} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{3}{2}} \right)^4.$$

(ii) Suppose $\beta_{23} \geq 1$ and $\beta_{14} \geq 1$. Then,

$$A_{n,\beta}^{(0)} \leq n^{-1} \sum_{|k_1|,|k_2|,|k_3| \leq n} |\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_1+k_3)\rho(k_1-k_2)|,$$

and this case can be treated as (7.31).

Case 2: Suppose that the graph is connected. This means that there is an edge connecting the sets V_1 and V_2. Suppose that $\beta_{\alpha_0\delta_0} \geq 1$, where $\alpha_0 \in \{1,2,3,4\}$ and $\delta_0 \in \{5,6,7,8\}$. We have then 7 nonzero coefficients β: β_{12}, β_{13}, β_{24}, β_{56}, β_{57}, β_{68} and $\beta_{\alpha_0\delta_0}$. Because all the edges have at least degree 2, there must be another nonzero coefficient β. Assume it is $\beta_{\alpha_1\delta_1}$. Then, the worse case will be when $\beta_{12} = \beta_{13} = \beta_{24} = \beta_{56} = \beta_{57} = \beta_{68} = \beta_{\alpha_0\delta_0} = \beta_{\alpha_1\delta_1} = 1$ and all the other coefficients are zero. Consider the change of variables $i_1 - i_2 = k_1$, $i_1 - i_3 = k_2$, $i_2 - i_4 = k_3$, $i_5 - i_6 = k_4$, $i_5 - i_7 = k_5$, $i_6 - i_8 = k_6$, $i_{\alpha_0} - i_{\delta_0} = k_7$. Then, it is easy to show that $i_{\alpha_1} - i_{\delta_1} = \mathbf{k} \cdot \mathbf{v}$, where $\mathbf{k} = (k_1, \ldots, k_5)$ and $\mathbf{v}$ is a 7-dimensional vector whose components are 0, 1 or -1. Then, using (A.2) and Hölder's inequality, we obtain

$$A_{n,\beta}^{(1)} \leq Cn^{-2} \left(\sum_{|k| \leq n} |\rho(k)| \right)^6 \leq Cn^{-2} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{3}{2}} \right)^4.$$

Proof of (7.32): We have

$$\sum_{q_1,\dots,q_8=2}^{\infty}\sum_{\beta\in\mathscr{C}_q^{(2)}} L_{q,\beta}^{(2)} = 9\mathrm{E}\left(\left|A(g'')^{(N)}(X_1)A(g_1')(X_1)A(g_1)(X_1)^2\right|^2\right)$$

$$\leq 9\|A(g'')^{(N)}\|^2_{L^8(\mathbb{R},\phi(x)dx)}\|A(g_1')^{(N)}\|^2_{L^8(\mathbb{R},\phi(x)dx)}\|A(g_1)^{(N)}\|^4_{L^8(\mathbb{R},\phi(x)dx)},$$

which converges as $N\to\infty$ to

$$9\|A(g'')\|^2_{L^8(\mathbb{R},\phi(x)dx)}\|A(g_1')\|^2_{L^8(\mathbb{R},\phi(x)dx)}\|A(g_1)\|^4_{L^8(\mathbb{R},\phi(x)dx)}<\infty.$$

Case $i=3$. The term $\mathrm{E}\left[[|\Phi_{n,N}^{(3)}|^2]\right]$ can be handled in a similar way and we omit the details. $\square$

7.3.3 Some other results

In [46] this estimate is obtained applying Poincare inequality to estimate the variance plus twice the integration-by-parts formula and for this reason one requires the function f to be four times differentiable.

Theorem 7.11.

(i) For functions $f\in\mathbb{D}^{4,4}(\mathbb{R},\phi(x)dx)$ with Hermite rank $d=2$, then (7.11).

(ii) For functions $f\in\mathbb{D}^{3d-2,4}(\mathbb{R},\phi(x)dx)$ with Hermite rank $d\geq 3$, then (7.12).

In [41] assuming only $f\in\mathbb{D}^{1,4}(\mathbb{R},\phi(x)dx)$ and applying Gebelein's inequality, instead of Poincare's inequality to estimate the variance of $\langle DF_n,u_n\rangle_{\mathfrak{H}}$ the authors have obtained the following weaker bound. This result applies to the case where $f(x)=|x|-\mathrm{E}[|Z|]$.

Theorem 7.12.

(i) For functions $f\in\mathbb{D}^{1,4}(\mathbb{R},\phi(x)dx)$ with Hermite rank $d=2$,

$$d_{\mathrm{TV}}(Y_n,Z)\leq\frac{C}{\sqrt{n}}.$$

(ii) For even functions $f \in \mathbb{D}^{1,4}(\mathbb{R}, \phi(x)dx)$,

$$d_{\mathrm{TV}}(Y_n, Z) \leq \frac{C}{\sqrt{n}} \left(\sum_{|k| \leq n} |\rho(k)| \right).$$

In [41], using a non-trivial combination of Gabelein's inequality and some new estimates involving Malliavin operators, authors obtained following result under minimal regularity assumptions. In particular, this result applies to the functions $f(x) = |x|^p - \mathrm{E}[|Z|^p]$ for any $p \geq 1$.

Theorem 7.13.

(i) For functions $f \in L^2(\mathbb{R}, \phi(x)dx) \cap \mathbb{D}^{1,4}(\mathbb{R}, \phi(x)dx)$ with Hermite rank $d = 2$,

$$d_{\mathrm{TV}}(Y_n, Z) \leq \frac{C}{\sqrt{n}} \left(\sum_{|k| \leq n} |\rho(k)| \right)^{1/2} + \frac{C}{\sqrt{n}} \left(\sum_{|k| \leq n} |\rho(k)|^{4/3} \right)^{3/2}.$$

It is also important to note the following lemma from [41] to conclude that under the assumptions Theorem 7.13 convergence in total variation holds if $\|\rho\|_{l^2} < \infty$.

Lemma 7.14. Let $\{\rho(k)\}_{k \in \mathbb{Z}} \in l^2$, and $0 < \alpha < 2$ and $\beta, \gamma > 0$ be such that

$$\frac{2-\alpha}{2} \leq \frac{\gamma}{\beta}.$$

Then,

$$\lim_{n \to \infty} \frac{1}{n^\gamma} \left(\sum_{|k| \leq n} |\rho(k)|^\alpha \right)^\beta.$$

7.4 Wasserstein distance

Theorem 7.15. For functions $f \in L^2(\mathbb{R}, \phi(x)dx)$ with Hermite rank $d = 2$,

$$d_{\mathrm{W}}(Y_n, Z) \leq \frac{C}{\sqrt{n}} \left(\sum_{|k| \leq n} |\rho(k)| \right)^{1/2} + \frac{C}{\sqrt{n}} \left(\sum_{|k| \leq n} |\rho(k)|^{3/2} \right)^2$$

provided $A(f) = \sum_{q=d}^{\infty} |a_q| H_q(x) \in \mathbb{D}^{2,6}(\mathbb{R}, \phi(x)dx)$.

Proof. Using the same ideas, we have the estimate

$$d_W(Y_n, Z) \leq C \sqrt{\operatorname{Var}\left[\langle D^2 V_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}} \right]} + C \mathrm{E}\left[(|\langle DV_n \otimes DV_n, F_n \rangle_{\mathfrak{H}^{\otimes 2}}| \right]. \tag{7.33}$$

Therefore, we need to estimate the quantities

$$\operatorname{Var}\left[\langle D^2 F_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}} \right] \text{ and } \mathrm{E}\left[|\langle DF_n \otimes DF_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}}| \right].$$

(i) Estimation of $\operatorname{Var}\left[\langle D^2 F_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}} \right]$. We will follow similar arguments as in the proof of Theorem 7.8. First, we write

$$\langle D^2 F_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}} = \frac{1}{n} \sum_{i,j=1}^{n} g''(X_i) g_2(X_j) \rho^2(i-j).$$

Using a limit argument, we obtain

$$\langle D^2 F_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}} = \lim_{N \to \infty} \Phi_{n,N},$$

where the convergence holds in $L^1(\Omega)$ and

$$\Phi_{n,N} = \frac{1}{n} \sum_{i,j=1}^{n} \sum_{q_1,q_2=2}^{N} c_{q_1} c_{q_2} q_1 (q_1 - 1) H_{q_1-2}(X_i) H_{q_2-2}(X_j) \rho^2(i-j).$$

Therefore, by Fatou's lemma

$$\operatorname{Var}\left[\langle D^2 F_n, v_n\rangle_{\mathfrak{H}^{\otimes 2}}\right] \le \liminf_{N\to\infty} \operatorname{Var}\left[\Phi_{n,N}\right].$$

We can write

$$\begin{aligned}\operatorname{Var}\left[\Phi_{n,N}\right] =& \frac{1}{n^2}\sum_{i_1,i_2,i_3,i_4=1}^{n}\sum_{q_1,q_2,q_3,q_4=2}^{N} q_1(q_1-1)q_3(q_3-1)c_{q_1}c_{q_2}c_{q_3}c_{q_4}\rho^2(i_1-i_2)\rho^2(i_3-i_4)\\ &\times \operatorname{Cov}(H_{q_1-2}(X_{i_1})H_{q_2-2}(X_{i_2}), H_{q_3-2}(X_{i_3})H_{q_4-2}(X_{i_4})).\end{aligned} \tag{7.34}$$

With a very similar calculation as in the proof of Theorem 7.8, we have

$$\operatorname{Cov}\left[H_{q_1-1}(X_{i_1})H_{q_2-1}(X_{i_2}), H_{q_3-1}(X_{i_3})H_{q_4-1}(X_{i_4})\right] = \sum_{\beta\in\mathscr{D}'_q} C_{q,\beta}\prod_{1\le j<k\le 4}\rho(i_j-i_k)^{\beta_{jk}}, \tag{7.35}$$

where $\mathscr{D}'_q$ is the set of nonnegative integers β_{jk}, $1\le j<k\le 4$, satisfying

$$q_\ell - 2 = \sum_{j\,\mathrm{or}\,k=\ell}\beta_{jk}, \quad \text{for} \quad 1\le \ell\le 4 \tag{7.36}$$

and

$$\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}\ge 1.$$

Substituting (7.35) into (7.34) yields

$$\begin{aligned}\operatorname{Var}\left[\Phi_{n,N}\right] =& \frac{1}{n^2}\sum_{i_1,i_2,i_3,i_4=1}^{n}\sum_{q_1,q_2,q_3,q_4=2}^{N}\sum_{\beta\in\mathscr{D}'_q} C_{q,\beta}\,q_1(q_1-1)q_3(q_3-1)c_{q_1}c_{q_2}c_{q_3}c_{q_4}\\ &\times \rho^{\beta_{12}+2}(i_1-i_2)\rho^{\beta_{13}}(i_1-i_3)\rho^{\beta_{14}}(i_1-i_4)\rho^{\beta_{23}}(i_2-i_3)\rho^{\beta_{24}}(i_2-i_4)\rho^{\beta_{34}+2}(i_3-i_4).\end{aligned}$$

Replacing $\beta_{12}+2$ and $\beta_{34}+2$ by β_{12} and β_{34}, the above equality can be rewritten as

$$\operatorname{Var}[\Phi_{n,N}] = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \sum_{q_1,q_2,q_3,q_4=2}^{N} \sum_{\beta \in \mathscr{E}_q} K_{q,\beta} c_{q_1} c_{q_2} c_{q_3} c_{q_4} \prod_{1 \le j<k \le 4} \rho(i_j - i_k)^{\beta_{jk}},$$

where

$$K_{q,\beta} = \frac{q_1!(q_2-2)!q_3!(q_4-2)!}{(\beta_{12}-2)!\beta_{13}!\beta_{14}!\beta_{23}!\beta_{24}!(\beta_{34}-2)!}$$

and $\mathscr{E}_q$ is the set of nonnegative integers β_{jk}, $1 \le j < k \le 4$, satisfying $\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24} \ge 1$, $\beta_{12} \ge 2$, $\beta_{34} \ge 2$ and

$$q_\ell = \sum_{j \,\mathrm{or}\, k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \le \ell \le 4.$$

We can write

$$\operatorname{Var}[\Phi_{n,N}] \le \sup_{\beta} A_{n,\beta} \sum_{q_1,q_2,q_3,q_4=2}^{N} \sum_{\beta \in \mathscr{E}_q} K_{q,\beta} |c_{q_1} c_{q_2} c_{q_3} c_{q_4}|,$$

where

$$A_{n,\beta} = \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \prod_{1 \le j<k \le 4} |\rho(i_j - i_k)|^{\beta_{jk}},$$

and the supremum is taken over all sets of nonnegative integers β_{jk}, $1 \le j < k \le 4$, satisfying $\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24} \ge 1$, $\beta_{12} \ge 2$, $\beta_{34} \ge 2$, for $1 \le j < k \le 4$ and

$$2 \le \sum_{j \,\mathrm{or}\, k=\ell} \beta_{jk}, \quad \text{for} \quad 1 \le \ell \le 4.$$

Then, in this case we have

$$A_{n,\beta} \le \frac{1}{n^2} \sum_{i_1,i_2,i_3,i_4=1}^{n} \left| \rho(i_1-i_2)^2 \rho(i_{\alpha_1}-i_{\alpha_2}) \rho(i_3-i_4)^2 \right|$$

where $\alpha_1 \in \{1,2\}$ and $\alpha_2 \in \{3,4\}$. After making the change $i_1 = i_1$, $k_1 = i_1 - i_2$, $k_2 = i_{\alpha_1} - i_{\alpha_2}$ and $k_3 = i_3 - i_4$, we obtain

$$A_{n,\beta} \le \frac{1}{n} \sum_{|k_i| \le n, i=1,2,3} \left| \rho(k_1)^2 \rho(k_2) \rho(k_3)^2 \right| \le \frac{C}{n} \sum_{|k| \le n} |\rho(k)|.$$

Now, it is left to show that

$$\sum_{q_1,q_2,q_3,q_4=2}^{N} \sum_{\beta\in\mathscr{E}_q} K_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}c_{q_4}| < \infty. \tag{7.37}$$

We have

$$\begin{aligned}
&\sum_{q_1,q_2,q_3,q_4=2}^{N} \sum_{\beta\in\mathscr{E}_q} K_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}c_{q_4}| = \sum_{q_1,q_2,q_3,q_4=2}^{N} q_1(q_1-1)q_3(q_3-1)|c_{q_1}c_{q_2}c_{q_3}c_{q_4}| \\
&\times \mathrm{E}\left[H_{q_1-2}(X_1)H_{q_2-2}(X_1)H_{q_3-2}(X_1)H_{q_4-2}(X_1)\right] \\
&= \mathrm{E}\left[(A(g'')^{(N)})^2(A(g_2)^{(N)})^2\right].
\end{aligned}$$

By Hölder's inequality, we obtain

$$\sum_{q_1,q_2,q_3,q_4=2}^{N} \sum_{\beta\in\mathscr{E}_q} K_{q,\beta}|c_{q_1}c_{q_2}c_{q_3}c_{q_4}| \le \|A(g'')^{(N)}\|_{L^4(\mathbb{R},\gamma)}^{1/2}\|A(g_2)^{(N)}\|_{L^4(\mathbb{R},\gamma)}^{1/2}.$$

From the hypothesis and the Proposition 7.18, $(A(g'')^{(N)})^2$ and $(A(g_2)^{(N)})^2$ converge to $A(g'')^2$ and $A(g_2)^2$ in $L^2(\mathbb{R},\gamma)$ respectively. Hence, (7.37) holds.

(ii) Estimation of $\mathrm{E}\left[|\langle DF_n\otimes DF_n, v_n\rangle_{\mathfrak{H}^{\otimes 2}}|\right]$. We can write

$$\langle DF_n\otimes DF_n, v_n\rangle_{\mathfrak{H}^{\otimes 2}} = n^{-\frac{3}{2}} \sum_{i,j,k=1}^{n} g'(X_i)g'(X_j)g_2(X_k)\rho(i-k)\rho(j-k).$$

We have, in the $L^1(\Omega)$ sense,

$$\langle DF_n, u_n\rangle_{\mathfrak{H}} = \lim_{N\to\infty} \Psi_{n,N},$$

where

$$\Psi_{n,N} = n^{-\frac{3}{2}} \sum_{i,j,k=1}^{n} \sum_{q_1,q_2,q_3=2}^{N} c_{q_1}c_{q_2}c_{q_3}q_1q_2H_{q_1-1}(X_i)H_{q_2-1}(X_j)H_{q_3-2}(X_k)\rho(i-k)\rho(j-k).$$

Therefore, by Fatou's lemma

$$\mathrm{E}\left[\langle DF \otimes DF, v\rangle^2_{\mathfrak{H}^{\otimes 2}}\right] \le \liminf_{N\to\infty} \mathrm{E}\left[\Psi^2_{n,N}\right].$$

We can write

$$\begin{aligned}
\mathrm{E}\left[\Psi^2_{n,N}\right] = n^{-3} \sum_{i_1,\dots,i_6=1}^{n} \sum_{q_1,\dots,q_6=2}^{N} \left(\prod_{i=1}^{6} c_{q_i}\right) q_1 q_2 q_4 q_5 \\
\times \mathrm{E}\left[H_{q_1-1}(X_{i_1}) H_{q_2-1}(X_{i_2}) H_{q_3-2}(X_{i_3}) H_{q_4-1}(X_{i_4}) H_{q_5-1}(X_{i_5}) H_{q_6-2}(X_{i_6})\right] \\
\times \rho(i_1-i_3)\rho(i_2-i_3)\rho(i_4-i_6)\rho(i_5-i_6).
\end{aligned} \tag{7.38}$$

Using Lemma 7.17, we obtain

$$\begin{aligned}
&\mathrm{E}\left[H_{q_1-1}(X_{i_1}) H_{q_2-1}(X_{i_2}) H_{q_3-2}(X_{i_3}) H_{q_4-1}(X_{i_4}) H_{q_5-1}(X_{i_5}) H_{q_6-2}(X_{i_6})\right] \\
&= \sum_{\beta\in\mathscr{D}_q} C_{q,\beta} \prod_{1\le j<k\le 6} \rho(i_j-i_k)^{\beta_{jk}},
\end{aligned} \tag{7.39}$$

where

$$C_{q,\beta} = \frac{(q_3-2)!(q_6-2)!\prod_{j=1,2,4,5}^{4}(q_j-1)!}{\prod_{1\le j<k\le 6}\beta_{jk}!}$$

and $\mathscr{D}_q$ is the set of nonnegative integers β_{jk}, $1\le j<k\le 6$, satisfying

$$\begin{aligned}
q_\ell - 1 &= \sum_{j \text{ or } k=\ell} \beta_{jk}, \quad \text{for} \quad \ell = 1,2,4,5\,, \\
q_3 - 2 &= \sum_{j \text{ or } k=3} \beta_{jk}, \\
q_6 - 2 &= \sum_{j \text{ or } k=6} \beta_{jk}.
\end{aligned} \tag{7.40}$$

Replacing (7.39) into (7.38) yields

$$\mathrm{E}(\Psi_{n,N}^2) = n^{-3} \sum_{i_1,\ldots,i_6=1}^{n} \sum_{q_1,\ldots,q_6=2}^{N} \sum_{\beta\in\mathscr{D}_q} C_{q,\beta} \left(\prod_{i=1}^{6} c_{q_i}\right) q_1q_2q_4q_5$$
$$\times \rho(i_1-i_3)\rho(i_2-i_3)\rho(i_4-i_6)\rho(i_5-i_6) \prod_{j,k=1,j<k}^{6} \rho(i_j-i_k)^{\beta_{jk}}.$$

Substituting $\beta_{13}+1$, $\beta_{23}+1$, $\beta_{46}+1$ and $\beta_{56}+1$ by β_{13}, β_{23}, β_{46} and β_{56}, respectively, we can write

$$\mathrm{E}(\Psi_{n,N}^2) = n^{-3} \sum_{i_1,\ldots,i_6=1}^{n} \sum_{q_1,\ldots,q_6=2}^{N} \sum_{\beta\in\mathscr{E}_q} K_{q,\beta} \left(\prod_{i=1}^{6} c_{q_i}\right) q_1q_2q_4q_5 \prod_{j,k=1,j<k}^{6} \rho(i_j-i_k)^{\beta_{jk}},$$

where

$$K_{q,\beta} = \frac{\beta_{13}\beta_{23}\beta_{46}\beta_{56}(q_3-2)!(q_6-2)!\prod_{j=1,2,4,5}^{4}(q_j-1)!}{\prod_{j,k=1,j<k}^{6}\beta_{jk}!}$$

and $\mathscr{E}_q$ is the set of nonnegative integers β_{jk}, $1\le j<k\le 6$, satisfying

$$q_\ell = \sum_{j\,\mathrm{or}\,k=\ell} \beta_{jk}, \quad \text{for} \quad \ell = 1,\ldots,6.$$

Hence

$$\mathrm{E}(\Psi_{n,N}^2) \le \sup_{\beta} A_{n,\beta} \sum_{q_1,\ldots,q_6=2}^{N} \sum_{\beta\in\mathscr{E}_q} K_{q,\beta} \left(\prod_{i=1}^{6} |c_{q_i}|\right) q_1q_2q_4q_5,$$

where

$$A_{n,\beta} = n^{-3} \sum_{i_1,\ldots,i_6=1}^{n} \prod_{1\le j<k\le 6} |\rho(i_j-i_k)|^{\beta_{jk}}$$

and the supremum is taken over all sets of nonnegative integers β_{jk}, $j,k=1,\ldots,6$, $j<k$, satisfying $\beta_{13}\ge 1$, $\beta_{23}\ge 1$, $\beta_{46}\ge 1$, $\beta_{56}\ge 1$ and

$$2 \le \sum_{j\,\mathrm{or}\,k=\ell} \beta_{jk}, \quad \text{for} \quad \ell = 1,\ldots,6. \tag{7.41}$$

As in the proof of Theorem 7.8, we can show that

$$\sum_{q_1,\dots,q_6=2}^{\infty}\sum_{\beta\in\mathscr{E}_q} K_{q,\beta}\left(\prod_{i=1}^{6}|c_{q_i}|\right) q_1q_2q_4q_5<\infty. \tag{7.42}$$

In fact,

$$\begin{aligned}
&\sum_{q_1,\dots,q_6=2}^{N}\sum_{\beta\in\mathscr{E}_q} K_{q,\beta}\left(\prod_{i=1}^{6}|c_{q_i}|\right) q_1q_2q_4q_5=\sum_{q_1,\dots,q_6=2}^{N}\left(\prod_{i=1}^{6}|c_{q_i}|\right) q_1q_2q_4q_5\\
&\times \mathrm{E}\left[H_{q_1-1}(X_1)H_{q_2-1}(X_1)H_{q_3-2}(X_1)H_{q_4-1}(X_1)H_{q_5-1}(X_1)H_{q_6-2}(X_1)\right]\\
&=\mathrm{E}[(A(g')^{(N)})^4(X_1)(A(g_2)^{(N)})^2(X_1)],
\end{aligned}$$

where, as before, $A(g')^{(N)}$ and $A(g_2)^{(N)}$ are the truncated expansions of $A(g')$ and $A(g_2)$, respectively. By Hölder's inequality, we can write

$$\sum_{q_1,\dots,q_6=2}^{N}\sum_{\beta\in\mathscr{E}_q} K_{q,\beta}\left(\prod_{i=1}^{6}|c_{q_i}|\right) q_1q_2q_4q_5\le \|A(g')^{(N)}\|_{L^6(\mathbb{R},\gamma)}^{\frac{2}{3}}\|A(g_2)^{(N)}\|_{L^6(\mathbb{R},\gamma)}^{\frac{1}{3}}.$$

From our hypothesis and in view of Proposition 7.18, $(A(g')^{(N)})^3$ and $(A(g_2)^{(N)})^3$ converge in $L^2(\mathbb{R},\gamma)$ to $A(g')$ and $A(g_2)$, respectively. Thus, (7.42) holds true.

To complete the proof, it remains to show that,

$$\sup_{\beta} A_{n,\beta}\le Cn^{-1}\left(\sum_{|k|\le n}|\rho(k)|^{\frac{3}{2}}\right)^4.$$

As in the proof of Theorem 7.8, in order to show this estimate we will make use of some ideas from graph theory. The exponents β_{jk} induce an unordered simple graph on the set of vertices $V=\{1,2,3,4,5,6\}$ by putting an edge between j and k whenever $\beta_{jk}\neq 0$. Because $\beta_{13}\ge 1$, $\beta_{23}\ge 1$, $\beta_{46}\ge 1$ and $\beta_{56}\ge 1$, there are edges connecting the pairs of vertices $(1,3)$, $(2,3)$, $(4,6)$ and $(5,6)$. Condition (7.41) means that the degree of each vertex is at least 2. Then we consider two cases, depending whether graph is connected or not.

Case 1: Suppose that the graph is not connected. This implies that $\beta_{12} \geq 1$, $\beta_{45} \geq 1$ and there is no edge between the sets $V_1 = \{1,2,3\}$ and $V_2 = \{4,5,6\}$. The worse case is when $\beta_{12} = \beta_{13} = \beta_{23} = \beta_{45} = \beta_{46} = \beta_{56} = 1$ and all the other exponents are zero. In this case we have the estimate

$$A_{n,\beta} \leq n^{-1} \left(\sum_{|k_1|,|k_2| \leq n} |\rho(k_1)\rho(k_2)\rho(k_1 - k_2)| \right)^2 .$$

Using (A.1), we obtain

$$A_{n,\beta} \leq Cn^{-1} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{3}{2}} \right)^4 .$$

Case 2: Suppose that the graph is connected. This means that there is an edge connecting the sets V_1 and V_2. Suppose that $\beta_{\alpha_0 \delta_0} \geq 1$, where $\alpha_0 \in \{1,2,3\}$ and $\delta_0 \in \{4,5,6\}$. We have then 5 nonzero coefficients β: β_{13}, β_{23}, β_{46}, β_{56} and $\beta_{\alpha_0 \delta_0}$. Because all the edges have at least degree 2, there must be at least two more nonzero coefficients β. Let us denote them by $\beta_{\alpha_1 \delta_1}$ and $\beta_{\alpha_2 \delta_2}$.

Then, the worse case will be when $\beta_{13} = \beta_{23} = \beta_{46} = \beta_{56} = \beta_{\alpha_0 \delta_0} = \beta_{\alpha_1 \delta_1} = \beta_{\alpha_2 \delta_2} = 1$ and all the other coefficients are zero. Consider the change of variables $i_1 - i_3 = k_1$, $i_2 - i_3 = k_2$, $i_4 - i_6 = k_3$, $i_5 - i_6 = k_4$, $i_{\alpha_0} - i_{\delta_0} = k_5$. Then, $i_{\alpha_1} - i_{\delta_1} = \mathbf{k} \cdot \mathbf{v}$ and $i_{\alpha_2} - i_{\delta_2} = \mathbf{k} \cdot \mathbf{w}$, where $\mathbf{k} = (k_1, \ldots, k_5)$ and $\mathbf{v}$, $\mathbf{w}$ are 5-dimensional linearly independent vectors whose components are 0, 1 or -1. Then, we can write, using (A.3) and Hölder's inequality,

$$\begin{aligned}
A_{n,\beta} &\leq n^{-2} \sum_{|k_i| \leq n, 2 \leq i \leq 5} \prod_{i=2}^{5} |\rho(k_i)| |\rho(\mathbf{k} \cdot \mathbf{v}) \rho(\mathbf{k} \cdot \mathbf{w})| \leq Cn^{-2} \left(\sum_{|k| \leq n} |\rho(k)| \right)^3 \\
&\leq Cn^{-1} \left(\sum_{|k| \leq n} |\rho(k)|^{\frac{3}{2}} \right)^4 .
\end{aligned}$$

□

Remark 7.16. In the case $g(x) = x^2 - 1$, the term $\mathrm{Var}\left[\langle D^2 F_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}} \right]$ is zero because $\langle D^2 F_n, v_n \rangle_{\mathfrak{H}^{\otimes 2}}$ is deterministic, and for the second term we get the estimate (7.6).

7.5 Technical results

Following we present some technical results with their proofs which are used in the previous sections. We first recall a formula for the expectation of the product of multiple stochastic integrals.

Lemma 7.17. Let $q_i \geq 1$ be integers, and consider functions $f_i \in \mathfrak{H}^{\odot q_i}$, $i = 1, \ldots, M$. Then,

$$\mathrm{E}\left[\prod_{i=1}^{M} I_{q_i}(f_i)\right] = \sum_{\beta \in \mathscr{D}_q} C_{q,\beta} \left(\otimes_{i=1}^{M} f_i\right)_\beta ,$$

where

$$C_{q,\beta} = \frac{\prod_{i=1}^{M} q_i!}{\prod_{1 \leq j < k \leq M} \beta_{jk}!},$$

$\mathscr{D}_q$ is the set of nonnegative integers β_{jk}, $1 \leq j < k \leq M$ satisfying

$$q_i = \sum_{j \,\mathrm{or}\, k = i} \beta_{jk}, \quad i = 1 \ldots, M ,$$

and $\left(\otimes_{i=1}^{M} f_i\right)_\beta$ denotes the contraction of β_{jk} indexes between f_j and f_k, for all $1 \leq j < k \leq M$.

Proof. The product formula for multiple stochastic integrals (see, for instance, [49, Theorem 6.1.1], or formula (2.1) in [5] for $M = 2$) says that

$$\prod_{i=1}^{M} I_{q_i}(f_i) = \sum_{\mathscr{P},\psi} I_{\gamma_1+\cdots+\gamma_M}\left(\left(\otimes_{i=1}^{M} f_i\right)_{\mathscr{P},\psi}\right), \tag{7.43}$$

where $\mathscr{P}$ denotes the set of all partitions $\{1, \ldots, q_i\} = J_i \cup \left(\cup_{k=1,\ldots,M, k \neq i} I_{ik}\right)$, where for any $i,k = 1, \ldots, M$, I_{ik} and I_{ki} have the same cardinality, ψ_{ik} is a bijection between I_{ik} and I_{ki} and $\gamma_i = |J_i|$. Moreover, $\left(\otimes_{i=1}^{M} f_i\right)_{\mathscr{P},\psi}$ denotes the contraction of the indexes ℓ and $\psi_{ik}(\ell)$ for any $\ell \in I_{ik}$ and any $i,k = 1 \ldots, M$. Then, the expectation $\mathrm{E}\left(\prod_{i=1}^{M} I_{q_i}(f_i)\right)$ corresponds to the case $\gamma_1 = \cdots = \gamma_M = 0$, and, if we specify the number of partitions for fixed cardinalities β_{jk}, we obtain the desired formula. □

In general, given a random variable $F \in L^2(\Omega, \mathfrak{F}, \mathrm{P})$ with chaos expansion (Theorem 2.13), the fact that $\mathrm{E}[|F|^p] < \infty$ for some $p > 2$ does not imply that the chaos expansion converges in $L^p(\Omega, \mathfrak{F}, \mathrm{P})$. The next proposition provides a partial result in this direction for $p = 2M$ and in the one-dimensional case, assuming that all the coefficients are nonnegative.

Proposition 7.18. Consider a function $g \in L^2(\mathbb{R}, \mathscr{B}(\mathbb{R})\phi(x)dx)$, with an expansion of the form $g(x) = \sum_{q=0}^{\infty} c_q H_q(x)$. Suppose that $c_q \geq 0$ for each $q \geq 0$ and $g \in L^{2M}(\mathbb{R}, \mathscr{B}(\mathbb{R})\phi(x)dx)$ for some $M \geq 1$. Consider the truncated sequence

$$g^{(N)} := \sum_{q=0}^{N} c_q H_q. \tag{7.44}$$

Then $(g^{(N)})^M$ converges in $L^2(\mathbb{R}, \mathscr{B}(\mathbb{R})\phi(x)dx)$ to g^M.

Proof. The proof will be done by induction on M. The result is clearly true for $M = 1$. Suppose that $M \geq 2$ and the result holds for $M - 1$. Using the product formula for Hermite polynomials, which is a particular case of (7.43), we can write

$$\begin{aligned} (g^{(N)})^M &= \sum_{q_1,\dots,q_M=0}^{N} \prod_{i=1}^{M} c_{q_i} H_{q_i} \\ &= \sum_{q_1,\dots,q_M=0}^{N} \left(\prod_{i=1}^{M} c_{q_i} \right) \sum_{(\beta,\gamma)\in\widehat{\mathscr{D}}_q} C_{q,\beta,\gamma} H_{\gamma_1+\cdots+\gamma_M} , \end{aligned}$$

where

$$C_{q,\beta,\gamma} = \frac{\prod_{i=1}^{M} q_i!}{\prod_{i=1}^{M} \gamma_i! \prod_{1\leq j<k\leq M} \beta_{jk}!} ,$$

and $\widehat{\mathscr{D}}_q$ is the set of nonnegative integers β_{jk}, $1 \leq j < k \leq M$ and γ_i, $1 \leq i \leq M$, satisfying

$$q_i = \gamma_i + \sum_{j \,\mathrm{or}\, k = i} \beta_{jk}, \quad i = 1, \dots, M . \tag{7.45}$$

As a consequence, we obtain

$$(g^{(N)})^M = \sum_{m=0}^{\infty} d_{m,N} H_m ,$$

where

$$d_{m,N} = \sum_{q_1,\dots,q_M=0}^{N} \left(\prod_{i=1}^{M} c_{q_i} \right) \sum_{(\beta,\gamma)\in\widehat{\mathscr{D}}_q, \gamma_1+\cdots+\gamma_M=m} C_{q,\beta,\gamma} \, .$$

The function g^M belongs to $L^2(\mathbb{R},\mathscr{B}(\mathbb{R})\phi(x)dx)$. Therefore, it will have an expansion of the form

$$g^M = \sum_{m=0}^{\infty} d_m H_m \, .$$

In order to compute the coefficients d_m, taking into account that $gH_m \in L^2(\mathbb{R},\mathscr{B}(\mathbb{R})\phi(x)dx)$ and, by the induction hypothesis, $(g^{(N)})^{M-1}$ converges to g^{M-1} in $L^2(\mathbb{R},\mathscr{B}(\mathbb{R})\phi(x)dx)$ as $N\to\infty$, we can write

$$d_m = \frac{1}{m!}\mathrm{E}\left[g^M H_m\right] = \lim_{N\to\infty}\frac{1}{m!}\mathrm{E}\left[g(g^{(N)})^{M-1}H_m\right].$$

To compute the expectation $\mathrm{E}\left[g(g^{(N)})^{M-1}H_m\right]$ we need the chaos expansion of $(g^{(N)})^{M-1}H_m$:

$$(g^{(N)})^{M-1}H_m = \sum_{q_1,\dots,q_{M-1}=0}^{N} \prod_{i=1}^{M-1} c_{q_i} \sum_{(\beta',\gamma')\in\widehat{\mathscr{D}}'_q} C_{q,\beta',\gamma'} H_{\gamma'_1+\cdots+\gamma'_M} \, ,$$

where

$$C_{q,\beta',\gamma'} = \frac{m!\prod_{i=1}^{M-1} q_i!}{\prod_{i=1}^{M}\gamma'_i!\prod_{1\le j<k\le M}\beta'_{jk}!} \, ,$$

and $\widehat{\mathscr{D}}'_q$ is the set of β's and γ's such that (7.45) holds for $i=1,\dots,M-1$ and

$$m = \gamma_M + \sum_{j\,\mathrm{or}\,k=M} \beta'_{jk} \, .$$

As a consequence,

$$\mathrm{E}\left[(g(g^{(N)})^{M-1}H_m\right] = \sum_{q=0}^{\infty} q!c_q \sum_{q_1,\dots,q_{M-1}=0}^{N} \prod_{i=1}^{M-1} c_{q_i} \sum_{(\beta'\gamma')\in\widehat{\mathscr{D}}'_q, \gamma'_1+\cdots+\gamma'_M=q} C_{q,\beta',\gamma'}$$

and, taking into account that the coefficients c_q are nonnegative and putting $q = q_M$,

$$d_m = \sum_{q_1,\dots,q_M=0}^{\infty} \prod_{i=1}^{M} c_{q_i} \sum_{(\beta',\gamma')\in\widehat{\mathscr{D}}'_q, \gamma'_1+\cdots+\gamma'_M=q_M} \frac{\prod_{i=1}^{M} q_i!}{\prod_{i=1}^{M} \gamma'_i! \prod_{1\le j<k\le M} \beta'_{jk}!} .$$

We claim that for any $(\beta',\gamma') \in \widehat{\mathscr{D}}'_q$ there exist a unique element $(\beta,\gamma) \in \widehat{\mathscr{D}}_q$ such that

$$\prod_{i=1}^{M} \gamma_i! \prod_{1\le j<k\le M} \beta_{jk}! = \prod_{i=1}^{M} \gamma'_i! \prod_{1\le j<k\le M} \beta'_{jk}! .$$

Indeed, it suffices to take $\beta_{jk} = \beta'_{jk}$ if $1 \le j < k \le M-1$, $\gamma_i = \beta'_{iM}$ for $i = 1,\dots,M-1$, $\gamma_M = \gamma'_M$, and $\beta_{jM} = \gamma'_j$ for $1 \le j \le M-1$. It follows that $\lim_{N\to\infty} d_{m,N} = d_m$. This implies that $(g^{(N)})^M$ converges in $L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(x)dx)$ to g^M and allows us to complete the proof. $\square$

Appendix A

Appendix

A.1 Some inequalities

We will give some particular versions of Brascamp-Lieb inequality in the following lemma which is obtained in the paper [46] using Brascamp-Lieb inequality. For the general inequality, one can consult the original paper by Brascamp-Lieb in [7].

Lemma A.1. Fix an integer $M \geq 2$. Let f be a non-negative function on the integers and set $\mathbf{k} = (k_1, \ldots, k_M)$. Then, we have:

(i) For any vector $\mathbf{v} \in \mathbb{R}^M$ whose components are 1 or -1

$$\sum_{\mathbf{k}\in\mathbb{Z}^M} f(\mathbf{k}\cdot\mathbf{v}) \prod_{j=1}^{M} f(k_j) \leq C \left(\sum_{k\in\mathbb{Z}} f(k)^{1+\frac{1}{M}} \right)^M . \tag{A.1}$$

(ii) For any vector $\mathbf{v} \in \mathbb{R}^M$ whose components are 0, 1 or -1, assuming $\sum_{k\in\mathbb{Z}} f(k)^2 < \infty$,

$$\sum_{\mathbf{k}\in\mathbb{Z}^M} f(\mathbf{k}\cdot\mathbf{v}) \prod_{j=1}^{M} f(k_j) \leq C \left(\sum_{k\in\mathbb{Z}} f(k) \right)^{M-1} . \tag{A.2}$$

(iii) Suppose $M \geq 3$. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^M$ be linearly independent vectors, whose components are 0, 1 or -1. Suppose $\sum_{k\in\mathbb{Z}} f(k)^2 < \infty$. Then,

$$\sum_{\mathbf{k}\in\mathbb{Z}^M} f(\mathbf{k}\cdot\mathbf{v}) f(\mathbf{k}\cdot\mathbf{w}) \prod_{j=1}^{M} f(k_j) \leq C \left(\sum_{k\in\mathbb{Z}} f(k) \right)^{M-2} . \tag{A.3}$$

See [20] for the proof of following extension of the Gronwall's lemma.

Lemma A.2. Let $f_n : [0,T] \to \mathbb{R}_+$ be a function such that

$$f_{n+1}(t) \leq \int_0^t f_n(s) g(t-s) ds$$

for all $t \in [0,T]$ and $n \in \mathbb{N}_0$, for a nonnegative function g which is integrable on $[0,T]$. Suppose that $f_0(t) \leq M$ for all $t \in [0,T]$. Then for all $n \in \mathbb{N}_0$ and $t \in [0,T]$,

$$f_n(t) \leq M a_n,$$

where $(a_n)_{n \in \mathbb{N}_0}$ is a sequence of positive numbers with the property that $\sum_n a_n^{1/p} < \infty$ for all $p \geq 1$. In particular, $\sum_n f_n(t)^{1/p}$ converges uniformly on $[0,T]$.

Remark A.3. Note that a_n's only depends on g not f_n's. More precisely, $a_n = G(T)^n \mathrm{P}(S_n \leq T)$ where $G(T) = \int_0^T g(s) ds$ and $S_n = \sum_{i=1}^n X_i$ where $(X_i)_{i \in \mathbb{N}}$ are i.i.d. random variables on $[0,T]$ with density $g(s)/G(T)$.

A.2 Brownian bridge

$\{\widehat{B}^{x,y}_{[a,b]}(s); s \in [a,b]\}$ denote a d-dimensional Brownian bridge in the time interval $[a,b]$ that goes from the starting point x at time a to the end point y at time b. We also set $\widehat{B}_{[a,b]} := \widehat{B}^{0,0}_{[a,b]}$. We recall that the Brownian bridge $\widehat{B}^{x,y}_{[a,b]}$ can be expressed as

$$\widehat{B}^{x,y}_{[a,b]}(s) = \widehat{B}_{[a,b]}(s) + \frac{s-a}{b-a} y + \frac{b-s}{b-a} x, \quad x, y \in \mathbb{R}^d. \tag{A.4}$$

Proposition A.4. Fix an integer $k \geq 2$. Let $B^j_{[a,b]}$, $j = 1, \ldots, k$ be independent d-dimensional Brownian bridges in $[a,b]$ from 0 to 0, where $[a,b] \subset [0,t]$. Consider a measurable function $\alpha =$

$\left(\alpha^{j,l}\right)_{1\leq j<l\leq k}:[a,b]\to\mathbb{R}^{k(k-1)/2}$. For each $1\leq j<l\leq k$ we set

$$G_{\varepsilon}^{j,l}:=\int_a^b\Lambda_{\varepsilon}(B^j_{[a,b]}(s)-B^l_{[a,b]}(s)+\alpha^{j,l}(s))ds.$$

Then the following results hold true:

(i) For each $\kappa\in\mathbb{R}$,

$$\sup_{\varepsilon\in(0,1]}\sup_{\alpha}\mathrm{E}\left[\exp\left(\kappa\sum_{1\leq j<l\leq k}G_{\varepsilon}^{j,l}\right)\right]=K_{t,\kappa}<\infty, \tag{A.5}$$

where the constant $K_{t,\kappa}$ only depends on t and κ.

(ii) For each $1\leq j<l\leq k$, the random variables $G_{\varepsilon}^{j,l}$ converge in $L^p(\Omega)$ for all $p\geq 2$, as $\varepsilon\downarrow 0$, to a limit denoted by $G^{j,l}:=\int_a^b\Lambda(B^j_{[a,b]}(s)-B^l_{[a,b]}(s)+\alpha^{j,l}(s))ds$.

(iii) We have, for all $\kappa\in\mathbb{R}$,

$$\lim_{\varepsilon\downarrow 0}\mathrm{E}\left(\exp\left(\kappa\sum_{1\leq j<l\leq k}G_{\varepsilon}^{j,l}\right)\right)=\mathrm{E}\left(\exp\left(\kappa\sum_{1\leq j<l\leq k}G^{j,l}\right)\right),$$

where the convergence is uniform in α and in a,b.

Proof. Property (A.5) has been proved in [26] (see Lemma 4.1 and the proof of (4.3) in Proposition 4.2 for details). For property (ii), in view of (A.5) it suffices to show that the convergence holds in $L^2(\Omega)$ as ε tends to zero. This is done by first showing that for any sequence $\varepsilon_k\downarrow 0$, the sequence of random variables $G_{\varepsilon_k}^{j,l}$ is Cauchy in $L^2(\Omega)$. This Cauchy property is established in proof of Proposition 4.2 in [26] for the case $\alpha^{j,l}(x)=x^j-x^l$ and the case of a general α can be done in a similar way. Finally, the convergence in (iii) is obtained by using properties (i) and (ii) and the elementary inequality $e^a-e^b\leq\frac{1}{2}(e^a+e^b)(a-b)$ for $a>b$. $\square$

A.3 Some elementary computations

Lemma A.5. For $0 < s < t$, $a, b \in \mathbb{R}^d$,

$$\boldsymbol{p}_{t-s}(a)\boldsymbol{p}_s(b) = \boldsymbol{p}_t(a+b)\boldsymbol{p}_{\frac{s(t-s)}{t}}(b - \frac{s}{t}(a+b)). \tag{A.6}$$

Lemma A.6. For $0 < r < s < t$ and $y, z, x \in \mathbb{R}$, we have

$$K_{r,z,s,y}(t,x) \le C_t \Phi_{r,z,s,y}(t,x),$$

where Φ and K are defined in (5.28) and (5.35) respectively.

Proof. Using the identity $p_t^2(a) = \frac{1}{\sqrt{2\pi t}} p_{t/2}(a)$, we see that the first term in $K_{r,z,s,y}(t,x)$ is bounded by a constant depending on t times the first term in $\Phi_{r,z,s,y}(t,x)$. So, we estimate the integral term in $K_{r,z,s,y}(t,x)$ that we denote by I. Using the above identity for the square of the Gaussian together with the identity (A.6) we get

$$\begin{aligned}
I &= \int_s^t \int_{\mathbb{R}} p_{t-\theta}^2(x-w) p_{\theta-s}^2(w-y) p_{\theta-r}^2(w-z)\, dw d\theta \\
&= \int_s^t \int_{\mathbb{R}} \frac{1}{\sqrt{(2\pi)^3 (t-\theta)(\theta-s)(\theta-r)}} p_{\frac{t-\theta}{2}}(x-w) p_{\frac{\theta-s}{2}}(w-y) p_{\frac{\theta-r}{2}}(w-z)\, dw d\theta \\
&= p_{\frac{t-s}{2}}(x-y) \int_s^t \int_{\mathbb{R}} \frac{1}{\sqrt{(2\pi)^3 (t-\theta)(\theta-s)(\theta-r)}} p_{\frac{(t-\theta)(\theta-s)}{2(t-s)}}(w - y - \frac{\theta-s}{t-s}(x-y)) \\
&\qquad \times p_{\frac{\theta-r}{2}}(w-z)\, dw d\theta.
\end{aligned}$$

Now, applying the semigroup property,

$$I = \frac{p_{\frac{t-s}{2}}(x-y)}{(2\pi)^{3/2}} \int_s^t \frac{1}{\sqrt{(t-\theta)(\theta-s)(\theta-r)}} p_{\frac{(t-\theta)(\theta-s)}{2(t-s)} + \frac{\theta-r}{2}}(z - y - \frac{\theta-s}{t-s}(x-y)) d\theta.$$

Since for $r < s < \theta < t$

$$\frac{\theta-r}{2} \le \frac{(t-\theta)(\theta-s)}{2(t-s)} + \frac{\theta-r}{2} \le \frac{t-r}{2},$$

we have

$$p_{\frac{(t-\theta)(\theta-s)}{2(t-s)}+\frac{\theta-r}{2}}(z-y-\frac{\theta-s}{t-s}(x-y)) \leq \frac{\sqrt{t-r}}{\sqrt{\theta-r}} p_{\frac{t-r}{2}}(z-y-\frac{\theta-s}{t-s}(x-y))$$

and

$$\begin{aligned} I &\leq \frac{p_{\frac{t-s}{2}}(x-y)}{(2\pi)^{3/2}} \int_s^t \frac{\sqrt{t-r}}{\sqrt{(t-\theta)(\theta-r)^2(\theta-s)}} p_{\frac{t-r}{2}}(z-y-\frac{\theta-s}{t-s}(x-y))d\theta \\ &\leq \frac{\sqrt{t-r}\, p_{\frac{t-s}{2}}(x-y)}{(2\pi)^{3/2}} J, \end{aligned}$$

where

$$J = \int_s^t \frac{\sqrt{t-r}}{(\theta-r)\sqrt{(t-\theta)(\theta-s)}} p_{\frac{t-r}{2}}(z-y-\frac{\theta-s}{t-s}(x-y))d\theta.$$

Making the change of variables $\frac{\theta-s}{t-s}=\gamma$ and putting $\beta=\frac{s-r}{t-s}>0$ yields $\theta-r=\theta-s+s-r=(t-s)(\gamma+\beta)$ and

$$J = \frac{1}{t-s}\int_0^1 (1-\gamma)^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}(\gamma+\beta)^{-1} p_{\frac{t-r}{2}}(z-y+\gamma(y-x))d\gamma.$$

We consider two cases:

Case 1: If $z-y$ and $z-x$ have same sign, then

$$p_{\frac{t-r}{2}}(z-y+\gamma(y-x)) \leq p_{\frac{t-r}{2}}(z-y)+p_{\frac{t-r}{2}}(z-x).$$

Case 2: If $z-y$ and $z-x$ have different sign, suppose firstly that $z-y>0$ and $z-x=z-y+y-x<0$. Then, $0<z-y<-(y-x)$; so $|z-y|<|y-x|$ and

$$p_{\frac{t-r}{2}}(z-y+\gamma(y-x)) \leq \frac{1}{\sqrt{\pi(t-r)}}\mathbf{1}_{\{|y-x|>|z-y|\}}.$$

Similarly, if $z-y<0$ and $z-x=z-y+y-x>0$, then $0>z-y>-(y-x)$, which implies $|z-y|<|y-x|$ and we end up with the same inequality.

Finally, noting that for $\beta=\frac{s-r}{t-s}>0$

$$\int_0^1(1-\gamma)^{-1/2}\gamma^{-1/2}(\gamma+\beta)^{-1}d\gamma=\frac{1}{\sqrt{\beta(\beta+1)}}=\frac{t-s}{\sqrt{(t-r)(s-r)}},$$

we get

$$\begin{aligned}
I &\le \frac{\sqrt{t-r}\,p_{\frac{t-s}{2}}(x-y)}{(2\pi)^{3/2}}J\\
&\le C_T\frac{p_{\frac{t-s}{2}}(x-y)}{\sqrt{s-r}}\left(p_{\frac{t-r}{2}}(z-y)+p_{\frac{t-r}{2}}(z-x)+\mathbf{1}_{\{|y-x|>|z-y|\}}\right)\\
&\le C_T'\frac{p^2_{t-s}(x-y)}{\sqrt{s-r}}\left(p^2_{t-r}(z-y)+p^2_{t-r}(z-x)+\mathbf{1}_{\{|y-x|>|z-y|\}}\right),
\end{aligned}$$

which then completes our proof by taking the square roots on both sides. □

Lemma A.7. Let Φ be as in (5.28) and $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. Then,

(a) For fixed $0<r<s<t$,

$$\int_{\mathbb{R}^2}\Phi_{r,z,s,y}(t,x)dydz\le C_t\left(1+\frac{1}{(s-r)^{1/4}}\right).$$

(b) For fixed $0<r<s<t$,

$$\int_{\mathbb{R}^2}\Phi^2_{r,z,s,y}(t,x)dydz\le\frac{C_t}{(s-r)^{1/2}(t-s)^{1/2}}\left(1+\frac{1}{(t-r)^{1/2}}\right),$$

and

$$\int_0^t\int_r^t\int_{\mathbb{R}^2}\Phi^2_{r,z,s,y}(t,x)dydzdsdr\le C_t,$$

Proof. Fix $0<r<s<t$ and $x\in\mathbb{R}$. (a) Using the semigroup property and Gaussian integrals, we

have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} p_{t-s}(x-y)\left(p_{s-r}(y-z)+\frac{p_{t-r}(z-y)+p_{t-r}(z-x)+\mathbf{1}_{\{|y-x|>|z-y|\}}}{(s-r)^{1/4}}\right)dydz\\
&=1+\frac{1}{(s-r)^{1/4}}+\frac{1}{(s-r)^{1/4}}\int_{\mathbb{R}^2} p_{t-s}(x-y)\mathbf{1}_{\{|y-x|>|z-y|\}}dydz\\
&\leq C_t\left(1+\frac{1}{(s-r)^{1/4}}\right).
\end{aligned}
$$

(b) Using $(a+b)^2 \leq 2(a^2+b^2)$,and $p_t^2(a)=\frac{1}{2\pi t}p_{t/2}(x)$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} p_{t-s}^2(x-y)\left(p_{s-r}^2(y-z)+\frac{p_{t-r}^2(z-y)+p_{t-r}^2(z-x)+\mathbf{1}_{\{|y-x|>|z-y|\}}}{(s-r)^{1/2}}\right)dydz\\
&=\frac{C_t}{(s-r)^{1/2}(t-s)^{1/2}}\left(1+\frac{1}{(t-s)^{1/2}}\right)\int_{\mathbb{R}^2} dydz p_{(t-s)/2}(x-y)\\
&\quad\times\left(p_{(s-r)/2}(y-z)+p_{(t-r)/2}(z-y)+p_{(t-r)/2}(z-x)+\mathbf{1}_{\{|y-x|>|z-y|\}}\right).
\end{aligned}
$$

Now, using semigroup property and integrating the last two term by elementary means, we obtain

$$
\int_{\mathbb{R}^2} \Phi_{r,z,s,y}^2(t,x)dydz \leq \frac{C_t}{(s-r)^{1/2}(t-s)^{1/2}}\left(1+\frac{1}{(t-r)^{1/2}}\right).
$$

Finally, after the change of variables $u=\frac{s-r}{t-r}$, the inner time integral becomes

$$
\int_r^t \frac{ds}{(s-r)^{1/2}(t-s)^{1/2}}=\int_0^1 \frac{du}{\sqrt{u(1-u)}}du=\pi
$$

and hence

$$
\int_0^t\int_r^t\int_{\mathbb{R}^2} \Phi_{r,z,s,y}^2(t,x)dydzdsdr \leq C_t\int_0^t\left(1+\frac{1}{(t-r)^{1/2}}\right)dr \leq C_t
$$

which completes the proof. □

Lemma A.8. Let F be a nonnegative random variable. Then $\mathrm{E}\left[F^{-p}\right]<\infty$ for all $p \geq 2$ if and only if for all $q \geq 2$, there exists $C=C(q)>0$ and $\varepsilon_0=\varepsilon_0(q)>0$ such that $\mathrm{P}(F<\varepsilon) \leq C\varepsilon^q$ for all

$\varepsilon \leq \varepsilon_0$.

Lemma A.9. Fix $t > 0$. Let $\phi_{R,t}$ and $\varphi_{R,t}$ be defined as in (6.1) and (6.38). Then, there exists $R_0 \geq 1$, depending on t, such that for all $0 < s < t$ and $R \geq R_0$:

(a) $c_t \leq \int_{\mathbb{R}} \phi^2_{R,t}(s,y)dy \leq C_t$, where the lower bound holds for $t/2 < s < t$.

(b) $\dfrac{c_t}{s \log R} \leq \int_{\mathbb{R}} \varphi^2_{R,t}(s,y)dy \leq \dfrac{C_t}{s \log R}$ where the lower bound holds for $t/2 < s < t$.

Proof. (a) We start with the upper bound. Using the semigroup property, we see that

$$\begin{aligned}
\int_{\mathbb{R}} \phi^2_{R,t}(s,y)dy &= \frac{1}{\sigma^2_{R,t}} \int_{Q_R^2} \int_{\mathbb{R}} p_{t-s}(y-x_1)p_{t-s}(x_2-y)dydx_1dx_2 \\
&= \frac{1}{\sigma^2_{R,t}} \int_{Q_R^2} p_{2(t-s)}(x_1-x_2)dx_1dx_2 \\
&\leq \frac{1}{\sigma^2_{R,t}} \int_{Q_R} \int_{\mathbb{R}} p_{2(t-s)}(x_1-x_2)dx_1dx_2 = \frac{2R}{\sigma^2_{R,t}} \leq C_t,
\end{aligned}$$

where the last bound follows from Lemma 6.1. To see the lower bound, let $R \geq 1$, and $t/2 < s < t$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} \phi^2_{R,t}(s,y)dy &= \frac{1}{\sigma^2_{R,t}} \int_{Q_R^2} p_{2(t-s)}(x_1-x_2)dx_1dx_2 \geq \frac{1}{2\sigma^2_{R,t}} \int_{Q^2_{R/\sqrt{2}}} p_{2(t-s)}(y_1)dy_1dy_2 \\
&\geq \frac{R}{\sqrt{2}\sigma^2_{R,t}} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} p_{2(t-s)}(y)dy \geq c_t,
\end{aligned}$$

where the last bound follows from Lemma 6.1.

(b) Similarly, using the semigroup property, we see that

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_{R,t}^2(s,y)dy &= \frac{1}{\Sigma_{R,t}^2} \int_{Q_R^2} \int_{\mathbb{R}} p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x_1) p_{\frac{s(t-s)}{t}}(y - \frac{s}{t}x_2) dy dx_1 dx_2 \\
&= \frac{1}{\Sigma_{R,t}^2} \int_{Q_R^2} p_{\frac{2s(t-s)}{t}}(\frac{s}{t}(x_1 - x_2)) dx_1 dx_2 \\
&= \frac{t^2}{s^2 \Sigma_{R,t}^2} \int_{Q_{sR/t}^2} p_{\frac{2s(t-s)}{t}}(y_1 - y_2) dy_1 dy_2 \\
&\leq \frac{2Rt}{s\Sigma_{R,t}^2} \leq \frac{C_t}{s \log R},
\end{aligned}$$

for all $R \geq R_0$, where the last bound follows from Lemma 6.6. To see the lower bound, let $t/2 < s < t$. Then, assuming $R \geq 1$,

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_{R,t}^2(s,y)dy &= \frac{t^2}{s^2 \Sigma_{R,t}^2} \int_{Q_{sR/t}^2} p_{\frac{2s(t-s)}{t}}(y_1 - y_2) dy_1 dy_2 \\
&\geq \frac{\sqrt{2}tR}{s\Sigma_{R,t}^2} \int_{Q_{\frac{sR}{t\sqrt{2}}}} p_{\frac{2s(t-s)}{t}}(z) dz \\
&\geq \frac{\sqrt{2}tR}{s\Sigma_{R,t}^2} \mathrm{P}\left(|N| \leq \frac{R}{2}\sqrt{\frac{s}{t(t-s)}} \right) \\
&\geq \frac{\sqrt{2}tR}{s\Sigma_{R,t}^2} \mathrm{P}\left(|N| \leq \frac{1}{2\sqrt{t}} \right) \geq \frac{c_t}{s \log R},
\end{aligned}$$

where the last bound follows from Lemma 6.6 and N denotes a $N(0,1)$ random variable. □

Lemma A.10. For all $R, t > 0$,

$$\int_{Q_R^2} p_t(x_1 - x_2) dx_1 dx_2 = \frac{4R}{\pi} \int_{\mathbb{R}} \varphi(\xi) e^{-t\frac{\xi^2}{R^2}} d\xi,$$

where

$$\varphi(\xi) = \frac{1 - \cos \xi}{\xi^2}.$$

Proof. See Appendix in [16]. □

Lemma A.11. For all $R \geq e$ and all $s > 0$,

$$\frac{1}{s}\int_0^s \frac{1}{r} e^{-s(\frac{s-r}{r})\frac{\xi^2}{R^2}} dr \leq 7 \log R \log(e + \frac{1}{s}) \log(e + \frac{1}{|\xi|}).$$

Proof. See [16, Lemma A.1]. □

Lemma A.12. Let $\{X_s : s \in [0,t]\}$ be a process such that $\sqrt{\mathrm{Var}[X_s]}$ is integrable on $[0,t]$. Then

$$\sqrt{\mathrm{Var}\left[\int_0^t X_s ds\right]} \leq \int_0^t \sqrt{\mathrm{Var}[X_s]} ds.$$

References

[1] Gideon Amir, Ivan Corwin, and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1+ 1 dimensions. *Communications on pure and applied mathematics*, 64(4):466–537, 2011.

[2] Raluca M Balan. A gentle introduction to spdes: the random field approach. *arXiv preprint arXiv:1812.02812*, 2018.

[3] Vlad Bally. *An elementary introduction to Malliavin calculus*. PhD thesis, INRIA, 2003.

[4] Fabrice Baudoin. *Diffusion processes and stochastic calculus*. European Mathematical Society, 2014.

[5] Denis Bell and David Nualart. Noncentral limit theorem for the generalized hermite process. *Electronic Communications in Probability*, 22:1–13, 2017.

[6] Hermine Biermé, Aline Bonami, Ivan Nourdin, and Giovanni Peccati. Optimal Berry-Esseen rates on the Wiener space: the barrier of third and fourth cumulants. *ALEA*, 9(2):473–500, 2012.

[7] Herm Jan Brascamp and Elliott H Lieb. Best constants in Young's inequality, its converse, and its generalization to more than three functions. *Advances in Mathematics*, 20(2):151–173, 1976.

[8] Peter Breuer and Péter Major. Central limit theorems for non-linear functionals of Gaussian fields. *Journal of Multivariate Analysis*, 13(3):425–441, 1983.

[9] María Emilia Caballero, Begoña Fernández, and David Nualart. Estimation of densities and applications. *Journal of Theoretical Probability*, 11(3):831–851, 1998.

[10] Le Chen. Moments, intermittency, and growth indices for nonlinear stochastic PDE's with rough initial conditions. Technical report, EPFL, 2013.

[11] Le Chen and Robert C Dalang. Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *The Annals of Probability*, 43(6):3006–3051, 2015.

[12] Le Chen and Jingyu Huang. Regularity and strict positivity of densities for the stochastic heat equation on $\mathbb{R}^d$. *arXiv preprint arXiv:1902.02382*, 2019.

[13] Le Chen and Kunwoo Kim. On comparison principle and strict positivity of solutions to the nonlinear stochastic fractional heat equations. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 53, pages 358–388. Institut Henri Poincaré, 2017.

[14] Le Chen, Yaozhong Hu, and David Nualart. *Regularity and Strict Positivity of Densities for the Nonlinear Stochastic Heat Equations*. AMS, American Mathematical Society, 2021.

[15] Le Chen, Davar Khoshnevisan, David Nualart, and Fei Pu. Central limit theorems for spatial averages of the stochastic heat equation via Malliavin–Stein's method. *Stochastics and Partial Differential Equations: Analysis and Computations*, pages 1–55, 2021.

[16] Le Chen, Davar Khoshnevisan, David Nualart, and Fei Pu. Spatial ergodicity for SPDEs via Poincaré-type inequalities. *Electronic Journal of Probability*, 26:1–37, 2021.

[17] Louis HY Chen, Larry Goldstein, and Qi-Man Shao. *Normal approximation by Stein's method*. Springer Science & Business Media, 2010.

[18] Daniel Conus. *The non-linear stochastic wave equation in high dimensions: existence, Hölder-continuity and Itô-Taylor expansion*. PhD thesis, Verlag nicht ermittelbar, 2008.

[19] Daniel Conus, Mathew Joseph, and Davar Khoshnevisan. On the chaotic character of the stochastic heat equation, before the onset of intermitttency. *The Annals of Probability*, 41 (3B):2225–2260, 2013.

[20] Robert Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous spde's. *Electronic Journal of Probability*, 4, 1999.

[21] Robert C Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A minicourse on stochastic partial differential equations*, volume 1962. Springer, 2009.

[22] Martin Hairer. An introduction to stochastic PDEs. *arXiv preprint arXiv:0907.4178*, 2009.

[23] Yaozhong Hu and David Nualart. Stochastic heat equation driven by fractional noise and local time. *Probability Theory and Related Fields*, 143(1):285–328, 2009.

[24] Yaozhong Hu, Fei Lu, and David Nualart. Convergence of densities of some functionals of Gaussian processes. *Journal of Functional Analysis*, 266(2):814–875, 2014.

[25] Yaozhong Hu, Jingyu Huang, David Nualart, and Samy Tindel. Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electronic Journal of Probability*, 20, 2015.

[26] Jingyu Huang, Khoa Lê, and David Nualart. Large time asymptotics for the parabolic Anderson model driven by spatially correlated noise. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 53,3, pages 1305–1340. Institut Henri Poincaré, 2017.

[27] Jingyu Huang, David Nualart, and Lauri Viitasaari. A central limit theorem for the stochastic heat equation. *Stochastic Processes and their Applications*, 130(12):7170–7184, 2020.

[28] Davar Khoshnevisan. *Analysis of stochastic partial differential equations*, volume 119. American Mathematical Soc., 2014.

[29] Markus Kunze. An introduction to Malliavin calculus. *Lecture notes*, 2013.

[30] Sefika Kuzgun and David Nualart. Rate of convergence in the Breuer–Major theorem via chaos expansions. *Stochastic Analysis and Applications*, 2019.

[31] Sefika Kuzgun and David Nualart. Convergence of densities of spatial averages of stochastic heat equation. *arXiv preprint arXiv:2108.09531*, 2021.

[32] Sefika Kuzgun and David Nualart. Feynman-Kac formula for iterated derivatives of the parabolic Anderson model. *Potential Analysis*, pages 1–23, 2022.

[33] Lars Larsson-Cohn. L^p-norms of Hermite polynomials and an extremal problem on Wiener chaos. *Arkiv för Matematik*, 40(1):133–144, 2002.

[34] Paul Malliavin. Stochastic calculus of variations and hypoelliptic operators. In *Proc. Internat. Symposium on Stochastic Differential Equations, Kyoto Univ., Kyoto,*. Wiley, 1978.

[35] Hiroyuki Matsumoto and Setsuo Taniguchi. *Stochastic analysis: Itô and Malliavin calculus in tandem*, volume 159. Cambridge University Press, 2016.

[36] Ivan Nourdin. Lectures on Gaussian approximations with Malliavin calculus. In *Séminaire de probabilités XLV*, pages 3–89. Springer, 2013.

[37] Ivan Nourdin and Giovanni Peccati. Stein's method on Wiener chaos. *Probability Theory and Related Fields*, 145(1-2):75–118, 2009.

[38] Ivan Nourdin and Giovanni Peccati. *Normal approximations with Malliavin calculus: from Stein's method to universality*, volume 192. Cambridge University Press, 2012.

[39] Ivan Nourdin and Giovanni Peccati. The optimal fourth moment theorem. *Proceedings of the American Mathematical Society*, 143(7):3123–3133, 2015.

[40] Ivan Nourdin, Giovanni Peccati, and Xiaochuan Yang. Berry-Esseen bounds in the Breuer-Major CLT and Gebelein's inequality. *Electronic Communications in Probability*, 24, 2019.

[41] Ivan Nourdin, David Nualart, and Giovanni Peccati. The Breuer–Major theorem in total variation: Improved rates under minimal regularity. *Stochastic Processes and their Applications*, 131:1–20, 2021.

[42] David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[43] David Nualart and Eulalia Nualart. *Introduction to Malliavin calculus*, volume 9. Cambridge University Press, 2018.

[44] David Nualart and Giovanni Peccati. Central limit theorems for sequences of multiple stochastic integrals. *The Annals of Probability*, 33:177–193, 2005.

[45] David Nualart and Lluís Quer-Sardanyons. Existence and smoothness of the density for spatially homogeneous SPDEs. *Potential Analysis*, 27(3):281–299, 2007.

[46] David Nualart and Hongjuan Zhou. Total variation estimates in the Breuer–Major theorem. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 57, pages 740–777. Institut Henri Poincaré, 2021.

[47] Eulalia Nualart. Lectures on Malliavin calculus and its applications to finance. *Lecture notes*, 2009.

[48] Etienne Pardoux. *Stochastic Partial Differential Equations: An Introduction*. Springer, 2021.

[49] Giovanni Peccati and Murad S Taqqu. *Wiener chaos: moments, cumulants and diagrams: a survey with computer implementation*, volume 1. Springer Science & Business Media, 2011.

[50] Nicolas Perkowski. SPDEs, classical and new. *Dimension*, 2:105, 2020.

[51] Nathan Ross. Fundamentals of Stein's method. *Probability Surveys*, 8:210–293, 2011.

[52] Marta Sanz-Solé. *Malliavin calculus with applications to stochastic partial differential equations*. EPFL press, 2005.

[53] Laurent Schwartz. *Théorie des distributions*, volume 2. Hermann Paris, 1966.

[54] Charles Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on*

Mathematical Statistics and Probability, Volume 2: Probability Theory. The Regents of the University of California, 1972.

[55] Ali S Üstünel. *An introduction to analysis on Wiener space*. Springer, 2006.

[56] John B Walsh. An introduction to stochastic partial differential equations. In *École d'Été de Probabilités de Saint Flour XIV-1984*, pages 265–439. Springer, 1986.